

EVALUATION AND EXTENSION OF LINEARIZED THEORIES
FOR TIRE MOTION AND WHEEL SHIMMY

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FIGURE

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Comparison of Damping Required for Stability

According to Approximations C, D1, D2 and
 D3 for a Sample Landing Gear Having the
 Properties $L = 0.8r$, $h = a = 0.5r$,

$$c_1 = c_2 = 0.25r, \epsilon = 0.3r, m_1 = 0.35m,$$

$$I_0 = mr^2, \text{ and } \tau = \rho = 0 \dots\dots\dots 159$$

LIST OF SYMBOLS

a	trail
a'	$a - gLh \sin \kappa / r$
A_1, A_2, A_3	coefficients defined by equation (7.5b)
B_1, B_2	coefficients defined by equation (7.5b)
c	lateral distance of center of pressure of vertical force from XZ-plane
c_λ	lateral shift of distance of center of pressure of vertical force from XZ-plane per unit of λ_0
c_γ	lateral shift of distance of center of pressure of vertical force from XZ-plane per unit of γ
c_1	distance from wheel center to center of gravity of swiveling parts of landing gear
c_2	distance from center of gravity of swiveling parts of landing gear to swivel axis
C_1	time constant
\underline{D}	tire parameter used by Bourcier de Carbon
D	differential operator with respect to distance, $\frac{d}{dx}$ or $v^{-1}D_t$
D_t	differential operator with respect to time, $\frac{d}{dt}$ or vD
E_0, E_1, \dots	coefficients of linear differential equations
f	frequency, $v/2\pi$
F_0, F_1, \dots	coefficients defined by equation (7.10b)

F_{yi}	lateral inertia force resulting from tire lateral deformation
F_{yn}	net lateral tire force acting on wheel
F_{ys}	net lateral structural force acting on wheel
$F_{y\lambda}$	lateral force acting on tire due to lateral distortion of tire
$F_{y\gamma}$	lateral force acting on tire due to lateral tilt of tire
F_z	vertical load on tire
F_{η}	lateral force of landing gear strut acting on the swiveling parts of the landing gear
g	linear damping constant (damping moment = $gD_t\dot{\psi}$)
h	half length of tire-ground contact area
I_0	moment of inertia of the swiveling part of a landing gear about an axis parallel to the swivel axis and passing through the center of gravity of the swiveling part
I_{xw}	polar moment of inertia of wheel and tire about an axis perpendicular to the wheel axle
I_{yt}	polar moment of inertia of tire
I_{yw}	total polar moment of inertia of wheel and tire
I_{ψ}	moment of inertia of the swiveling part of a landing gear about the swivel axis
k	lateral spring constant of landing gear strut
K_a	torsional stiffness of tire

ΔK_a	total effective change in tire torsional stiffness due to tire inertia effects
ΔK_{a1}	effective change in tire torsional stiffness due to tire lateral acceleration
ΔK_{aj}	change in tire torsional stiffness due to centrifugal forces
K_Y	lateral tire force due to tilt per radian tilt angle
K_λ	lateral stiffness of tire
ΔK_λ	total effective change in tire lateral stiffness due to tire inertia effects
$\Delta K_{\lambda 1}$	effective change in tire lateral stiffness due to tire lateral acceleration
$\Delta K_{\lambda j}$	change in tire lateral stiffness due to centrifugal forces
l_0, l_1, \dots	tire constants, $l_n = (nL + h) h^{n-1}/n!$
L	relaxation length
m	mass of swiveling parts of landing gear
m_t	mass of tire
m_w	mass of wheel including tire
m_l	mass of nonswiveling parts of landing gear
M_{xs}	net structural tilting moment acting on wheel center
$M_{x\theta}$	gyroscopic moment due to swiveling

M_{z1}	inertia moment resulting from tire lateral deformation
M_{zs}	net structural swiveling moment acting on wheel center
M_{za}	torsional moment acting on tire due to twist of tire
$M_{z\gamma}$	gyroscopic moment due to tilting
$M_{z\lambda}$	gyroscopic moment due to lateral tire distortion
M_{\downarrow}	damping moment about swivel axis
N	cornering power, lateral tire force per radian yaw angle during steady yawed rolling for yaw angle approaching zero
p_1, p_2	functions defined in appendix
$p_{1\infty}, p_{2\infty}, p_{11}, p_{12}, p_{22}$	functions defined in and after equations (5.2)
r	free tire radius
r_g	tire polar radius of gyration
r_3	vertical distance from wheel axle to ground
r_4	tire torus radius
\underline{R}	tire parameter used by Bourcier de Carbon
s	circumferential coordinate on tire
S	wave length, $\frac{2\pi}{v_1}$
\underline{S}	tire parameter used by Bourcier de Carbon
t	time

T	tire parameter used by Bourcier de Carbon
T_1, T_2, \dots	functions of D_t correlating structural forces, moments and deflections
u_κ	$c_\lambda F_z (\sigma l_1 \cos \kappa - a) \sin \kappa$
$u_{\kappa 1}$	$c_\lambda F_z (l_1 \cos \kappa - a) \sin \kappa$
v	rolling velocity
w	width of tire-ground contact area
w_1	density
w_κ	$(aK_\gamma + aF_z + c_\gamma F_z \sin \kappa) \sin \kappa$
X, Y, Z	coordinate axes; X-axis is a space fixed horizontal axis parallel to the mean direction of rolling motion, Z-axis is vertical, and Y-axis is perpendicular to the XZ-plane
z	vertical distance up from XY ground plane
y_0, y_1, y_2, y_l, y_g	lateral deflection of tire equator from XZ-plane; subscript o refers to the center of the ground contact area, l to the foremost point of the ground contact area, 2 to the rearmost point of the ground contact area, l to equator points off the ground and g to equator points on the ground
α	twist in tire, radians
γ	lateral wheel tilt, radians
γ_λ	lateral tire tilt resulting from lateral deformation, radians

- $\epsilon, \underline{\epsilon}$ pneumatic caster, K_a/N
- $\eta, \eta_0, \eta_3, \eta_a, \eta_1, \eta_g$ lateral deflection of wheel center plane with respect to the XZ-plane; subscript 0 refers to the point corresponding to the center of the ground contact area, 3 to the center point of the wheel, a to the point of attachment of the swiveling parts of the wheel to the swivel axis, 1 to wheel plane points off the ground and g to wheel plane points on the ground
- η_y inertia force parameter
- η_z inertia moment parameter
- θ rotation of wheel about the vertical Z-axis, radians
- κ inclination of swivel axis, radians (See Figure 5)
- $\lambda, \lambda_0, \lambda_1, \lambda_2, \lambda_i, \lambda_g$ lateral distortion of tire equator with respect to the solid parts of the wheel; subscript 0 refers to the center of the ground contact area, 1 to the foremost point of the ground contact area, 2 to the rearmost point of the ground contact area, i to equator points off the ground and g to equator points on the ground
- ω circular frequency of shimmy motion, $2\pi f$ or $\omega_1 v$

ν_1	path frequency of shimmy motion, νv^{-1}
ξ	tire tilt parameter
ρ	spring constant for a linear restoring moment
ρ_κ	$(aK_Y - aF_Z + ac_\lambda F_Z + c_Y F_Z \sin \kappa) \sin \kappa$
$\rho_{\kappa 1}$	$-aF_Z \sin \kappa \cos \kappa$
σ	$1 + \xi Lh \tan \kappa / l_1 r$
σ_1, σ_2	constants representing phase shift
τ	constant defined by equations (3.13) and (3.30)
τ_1	constant for gyroscopic moment $\left(\approx \frac{1}{2}\right)$
τ_2	$N l_1 C_1^2 / I_\psi$
ψ	rotation of wheel about the swivel axis, radians
ω	angular velocity of wheel about its axle $\left(\approx \frac{v}{r}\right)$

Subscripts:

c	critical
m	maximum

CHAPTER I

INTRODUCTION

In the ground maneuvering of aircraft equipped with swiveling landing gears, there sometimes arises the problem of violent oscillations or shimmy of the landing gear which may lead to failure of the gear. In the past this problem has been handled largely by the empirical procedure of equipping landing gears with supplementary shimmy dampers whose sizes have been controlled largely by practical experience. However, this empirical type of approach has not proved entirely satisfactory as is evidenced by occasional difficulties which are experienced with wheel shimmy. Moreover, for radically different types of complex flexible landing gears it is highly doubtful whether any empirical approach based purely on past experience could always safely and optimally take into account all of the possible conditions which a landing gear might be subjected to in actual operation.

Historical Background

Because of these considerations a considerable amount of theoretical and experimental work on wheel shimmy has been done, mostly in the past 25 years. The historical

background of this work, as taken in part from a paper by Dengler, Goland, and Herrman¹ may be briefly described as follows. Wheel shimmy first arose as a problem in automobiles around the year 1920 and from that time until the midthirties a considerable amount of research was devoted to this automobile problem. Much of this early research was concerned with factors peculiar to the automobile problem and is not directly applicable to the aircraft problem which is of primary concern in the present investigation. However, two important fundamental contributions to an understanding of the general wheel shimmy problem were made in this period by Broulhiet² in France in 1925 and by Fromm³ in Germany several years later. These two investigators were apparently the first ones to recognize the importance of tire lateral flexibility and cornering power as primary factors influencing the occurrence of wheel shimmy.

In the midthirties the aircraft wheel shimmy problem became of importance and most of the subsequent literature

¹ Max Dengler, Martin Goland, and Georg Herrman, "A Bibliographic Survey of Automobile and Aircraft Wheel Shimmy," WADC Technical Report 52-141, 1951, 142 pp.

² M. G. Broulhiet, "La Suspension de la Direction de la Voiture Automobile, Shimmy et Dandinement," Bull. Soc. Ing. Civ., Vol. 78, July 1925, Pp. 540-554.

³ H. Fromm, "Kurzer Berichte über die Geschichte des Theorie des Radflatterns," Berichte 140 der Lilienthal-Gesellschaft, 1941, Pp. 53-56.

on wheel shimmy is concerned with the aircraft problem. At the beginning of this period, a number of significant contributions were made by Schlippe and Dietrich,^{4,5,6} Melzer,⁷ and Maier⁸ in Germany, Greidanus⁹ in the Netherlands, Kantrowitz¹⁰ and Wylie¹¹ in the United States, and Temple¹² and Taylor¹³ in England. These various investigators each

⁴ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," Berichte 140 der Lilienthal-Gesellschaft, 1941, Pp. 35-45, 63-66.

⁵ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, ZWB Special Publication, 1942, 20 pp.

⁶ B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," ZWB Technische Berichte, Vol. 11, No. 2, 1944, Pp. 1-16.

⁷ M. Melzer, "Beitrag zur Theorie des Spornradflatterns," ZWB Technische Berichte, Vol. 7, No. 2, 1940, Pp. 59-70.

⁸ E. Maier, "Theoretische Untersuchungen über die Stabilität von Flugzeugfahrwerken," ZWB FB 1166, 1940, 59 pp.

⁹ J. H. Greidanus, "Control and Stability of the Nose-Wheel Landing Gear," Report V 1038, Netherlands National Aeronautical Research Institute, 1942, 27 pp.

¹⁰ Arthur Kantrowitz, "Stability of Castering Wheels for Aircraft Landing Gears," NACA Technical Report 686, 1940, 16 pp.

¹¹ Jean Wylie, "Dynamic Problems of the Tricycle Alighting Gear," Journal of the Aeronautical Sciences, Vol. 7, No. 2, Dec. 1939, Pp. 61-67.

¹² G. Temple, "Preliminary Report on the Theory of Shimmy in Aeroplane Nose Wheels and Tail Wheels," RAE Report No. AD 3148, 1940, 40 pp.

¹³ J. Lockwood Taylor, "Oscillation of Castering Wheels," Aircraft Engineering, Vol. 13, No. 143, Jan. 1941, p. 13.

developed at least slightly different theories of tire motion and wheel shimmy, most of which take tire elasticity into account in different ways. Also experimental contributions were furnished by most of these investigators and also by Dietz and Harling¹⁴ and Schrode.¹⁵ However, as yet, no thorough evaluation has been made of these various theories and data to determine the absolute and relative merits of the theories.

The major recent contributions to the wheel shimmy problem are the work of Rotta,¹⁶ Bourcier de Carbon,¹⁷ and Moreland.^{18,19} Rotta slightly extended the most advanced earlier theory, developed by Schlippe and Dietrich, and made

¹⁴ O. Dietz and R. Harling, "Experimentelle Untersuchungen über das Spornradflattern," ZWB FB 1320, 1940, 101 pp.

¹⁵ H. Schrode, "Verminderung der Flatterneigung von Sporn- und Bugwerken durch Einbau besonders geformter Reifen," ZWB Technische Berichte, Vol. 10, No. 4, April 1943, Pp. 113-116.

¹⁶ J. Rotta, "Properties of the Aeroplane During Take-Off and Alighting," Part 1: Reports and Translations No. 969, Dec. 1947, 63 pp; Part 2: Reports and Translations No. 970, Feb. 1948, 85 pp., British Ministry of Supply.

¹⁷ Christian Bourcier de Carbon, "Étude Théorique du Shimmy des Roues d'Avion," Office National d'Etudes et de Recherches Aéronautiques, Publication No. 7, 1948, 98 pp.

¹⁸ William J. Moreland, "Landing-Gear Vibration," AF Technical Report No. 6590, 1951, 70 pp.

¹⁹ William J. Moreland, "The Story of Shimmy," Journal of the Aeronautical Sciences, Vol. 21, No. 12, Dec. 1954, Pp. 793-808.

a detailed study of many of the fundamental properties of pneumatic tires which enter into the wheel shimmy problem. Bourcier de Carbon developed a theory of wheel shimmy much like the earlier theory of Greidanus which, although not of so advanced a nature as the Schlippe-Dietrich or Rotta theories, is perhaps the most complex existing theory which would be acceptable to aircraft designers from practical considerations. Bourcier de Carbon also pointed out some of the limitations of the earlier theories of Kantrowitz and Wylie. Moreland has advanced the idea that it may be more important for many shimmy problems to take into account landing gear structural elasticity rather than to take tire elasticity into account in great detail and he has developed several versions of a simplified tire motion theory.

Purpose of the Present Investigation

It is evident from the preceeding historical background that there exist in the literature a large number of theoretical and experimental papers dealing with the subject of wheel shimmy. However, most of these theoretical papers have not been correlated with each other or with the available experimental data so that consequently there exist at present a large number of at least superficially different theories of wheel shimmy and a fair amount of experimental

data which have not been correlated with many of these theories. The primary purpose of the present investigation is to clear up this partial confusion of theories by demonstrating that most of the previously published theories represent various approximations to one basic general linearized theory derived herein and that most of the previously published linearized theories which do not represent approximations to this general theory possess certain undesirable characteristics. This basic general theory, which is henceforth called the summary theory, is derived in such a manner that it makes use of and is compatible with the soundest features of practically all of the previously published theories, insomuch as this is possible at present; however, for the main part, this summary theory is a minor modification of the theory proposed by Schlippe and Dietrich.^{20,21,22}

A second purpose of this paper is to develop a series of systematic approximations to the summary theory suitable for use in the treatment of problems too simple to merit the use of the complete summary theory and to assess both these systematic approximations and the previously published

²⁰ B. von Schlippe and R. Dietrich, "Das Flattern eines bepunkteten Rades," op. cit.

²¹ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

²² B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

theories both with respect to how these theories are related to the summary theory and with respect to how the predictions of these theories agree with the experimental data available.

Statement of the Problem Considered and
the General Approach Thereto

The purpose of the next section of this chapter is to specifically define the problem considered in this thesis and to clarify in detail the correlation between the various parts of the thesis which deal with different aspects of this same problem.

The basic problem to be considered herein is the rolling motion and wheel shimmy of a rigid wheel equipped with an elastic tire where the wheel is attached to some supporting structure such as a landing gear strut. The motion of the rigid wheel can, of course, be completely described by six independent variables corresponding to the three degrees of freedom in translation and rotation of the wheel. In addition to these six degrees of freedom, there exists a seventh degree of freedom which is associated with the distortion of the elastic tire or the track of the tire on the ground which results from the application of a given motion to the rigid wheel. Thus, in general, the motion of a rigid wheel with an elastic tire represents a system of motion involving seven variables and consequently to solve

for the motion of a landing gear under arbitrary rolling conditions, seven equations correlating these different variables are required. Six of these equations will usually be the equations expressing the sum of the forces or moments acting along each of the three principal coordinate axes; the seventh relation will be an equation, usually a kinematic equation, which correlates the tire distortion with the other variables.

The present paper will not be concerned with all of these seven degrees of freedom. Specifically, the major part of this paper will be restricted to a consideration of cases of wheel motion where the wheel is rolling at an approximately constant velocity v without braking and consequently with constant angular velocity ω and where no strong vertical oscillations are involved. Thus, for example, effects of acceleration or deceleration, which are known to have at least some influence on the rolling motion,²³ are not considered. Similarly fore and aft oscillations of the wheel are excluded.

With the above three restrictions, the seven variable problem of a rolling wheel becomes reduced to the consideration of a system involving the following four degrees of freedom: (1) swiveling of the wheel about a vertical axis through the wheel center point, designated by the symbol θ ;

²³ H. Schrode, op. cit.

(2) lateral tilting of the wheel with respect to a vertical plane parallel to the direction of undisturbed motion, designated by the symbol γ ; (3) lateral displacement of the wheel with respect to a reference space fixed axis parallel to the direction of undisturbed motion, designated by the symbol η ; and (4) the lateral displacement of the tire footprint on the ground (which is a measure of the tire distortion), designated by the symbol y_0 . These coordinates and their positive directions are shown in Figure 1.

In order to obtain four equations correlating these four variables, θ , γ , η and y_0 , the following procedure is followed in the present paper. First of all, after some remarks on general restrictions, a derivation is given in Chapter II to establish a kinematic equation relating the four variables. Then, in Chapter III, the primary forces and moments acting on a rolling elastic wheel are discussed and are used later in Chapter IV to establish the other three equations.

After having established these general equations of motion, it is recognized that for many applications these equations in their most general form are relatively complicated and, while they are not by any means insoluble in this general form, it is, however, profitable to simplify the equations for those problems which do not require the detailed equations of the summary theory. Therefore, a number of

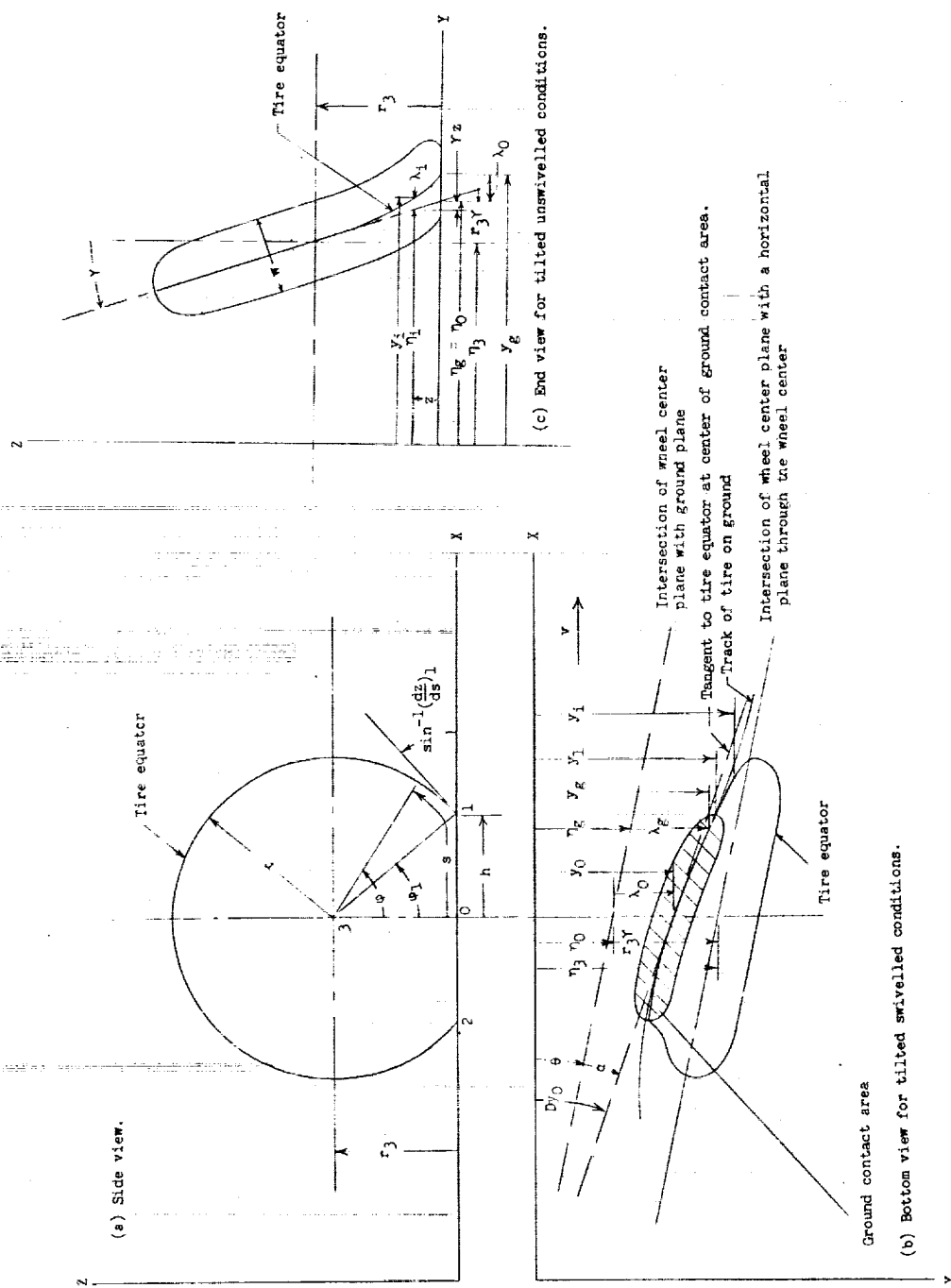


FIGURE 1
GEOMETRICAL RELATIONSHIPS FOR A ROLLING ELASTIC TIRE

of systematic approximations to the summary theory are formulated in Chapter V. A second reason for establishing these systematic approximations lies in the fact that they furnish a framework for comparing the summary theory with the other existing theories of wheel motion, most of which are closely related to these systematic approximations. Such a comparison of the summary theory and its systematic approximations with the existing theories of wheel motion is carried out in Chapter VI.

In Chapter VII, which is the last major part of this thesis, the application of the summary and approximate theories to two cases of simplified types of landing gear configurations is discussed. The first case is chosen primarily to demonstrate the correlation between theory and experiment and the second case to demonstrate the correlation between the summary and systematic approximation theories.

General Restrictions

Before entering upon the detailed derivation of the equations of motion, some mention will be made here as to some, as yet not discussed, restrictions on the analysis to follow. First of all, the present paper is limited exclusively to the subject of linearized theories. Some mention should be made here, however, as to the question of whether

a linearized theory is sufficient to describe the important features of wheel shimmy. In regard to this question, it appears at present that a linearized theory will provide at least a fair qualitative description of shimmy behavior in regard to the location of stability boundaries and to the question as to whether a given motion is stable or not. However, agreement between theory and experiment, to be shown later in this paper, is still not good enough from a quantitative point of view to warrant the conclusion that nonlinear effects can always be neglected or replaced by equivalent linear effects.

Another limitation of the linearized theory is that it does not permit calculation of the maximum steady state shimmy amplitude for those steady state self-excited shimmy motions which sometimes occur on actual landing gears.

While the preceding considerations suggest that nonlinear effects in landing gear motions may possibly be of importance for some practical problems, their consideration will be considered beyond the scope of the present investigation and henceforth only linearized theory is discussed.

Another restriction on the considerations of this investigation rises in connection with the assumption adopted throughout this paper that the finite width of a tire need not be taken into account in developing a tire motion theory for single tires of conventional cross section. This

assumption appears at present to be at least partly justified on the basis of the experiments of Schlippe and Dietrich;²⁴ on the other hand, since their tests relevant to this matter were extremely limited in scope, it is conceivable that their experimental results may not be completely typical. Consequently, perhaps it would be well if some future attention were directed to a more thorough evaluation of tire width effects. While this question will be considered beyond the scope of the present investigation, it is noted that some theoretical work on this subject has been already done by Schlippe and Dietrich²⁵ and later by Rotta.²⁶

²⁴ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

²⁵ Ibid.

²⁶ J. Rotta, op. cit.

CHAPTER II

KINEMATIC RELATIONSHIPS FOR THE ROLLING TIRE

In this chapter, the kinematic equations for the motion of a rolling tilted elastic tire without skidding are derived in accordance with the theoretical analysis of Schlippe and Dietrich.²⁷ This derivation differs only slightly from that analysis in that it omits some refinements of the theory which are necessary for very wide tires and it includes some influences of tilting of the tire in more detail. It should be noted here, however, that the modifications made in regard to tilt may not be necessarily of any great practical importance for most problems; however, since a few problems are conceivable where they may be of interest, they are included.

Specificly, the object of this chapter is to obtain a relation correlating the absolute lateral deflection of the center point of the tire ground contact area y_0 with the corresponding wheel coordinates of lateral deflection η , swivel angle θ and tilt γ . (These coordinates are illustrated in Figure 1.) To attain this object, the following procedure is used. First, some necessary geometrical relations are set down and some necessary background information regarding tire distortion is discussed. Then this information

²⁷ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

is utilized to obtain a kinematic relation between the lateral deflection of the tire center band or equator at the leading edge of the ground contact area and the coordinates η , γ and θ . Next a kinematic relation between the lateral deflections of the equator at the center and leading edge of the ground contact area (designated by y_0 and y_1 respectively) is established and finally these two relations are combined to obtain a basic kinematic equation correlating y_0 with η , γ and θ .

The derivation of these kinematic relations is based upon the following physical concepts. As a tire moves forward, the tire material on the circumference just ahead of the ground contact area is laid down or developed on the ground without skidding with respect to the ground to become the new leading portion of the ground contact area, so that the track of the tire is completely determined by the lateral distortion coordinate of the foremost ground contact point y_1 and the slope of the distorted centerline or equator of the tire at that point.

Geometrical Relationships

The primary geometrical quantities involved in the problem of a rolling tire are shown in Figure 1, which gives an instantaneous view of a distorted tire with respect to an arbitrary space-fixed XYZ coordinate system, the X-axis

being parallel to the ground and approximately parallel to the direction of wheel motion, the Z-axis being perpendicular to the ground, and the Y-axis being perpendicular to the X- and Z-axes. Parts (a) and (b) of this Figure represent side and bottom views, respectively, for a rolling wheel under swiveled tilted conditions. For the sake of clarity, part (c) of this Figure, which shows an end view of the rolling tire, has been drawn to a different scale from part (b) of this Figure and also has been drawn for the unswiveled condition only. In discussing the geometrical quantities in this Figure, the following terminology and symbols are used. By the wheel center plane is meant the plane of symmetry of the wheel perpendicular to the wheel axle. By the tire center-line or equator is meant the tire points which on the undistorted tire are located at the intersection of the tire outer circumference with the wheel center plane; under the action of moments and lateral forces these tire points are deflected laterally by an amount λ with respect to the wheel center plane. The symbol λ_1 designates the lateral deflection of tire equator points which are not in contact with the ground and λ_g designates the lateral deflection of points which are in contact with the ground. The particular ground contact point at the center of the ground contact area is designated by λ_0 .

The lateral distance of the wheel plane from an arbitrary space-fixed XZ plane is designated as η_1 for points off the ground at a vertical height z and as η_g for points on the ground. The lateral distance of tire equator points from this XZ-plane are similarly designated as y_1 and y_g , respectively. The difference between y and η is the tire lateral distortion λ or

$$\lambda_1 = y_1 - \eta_1 \quad (2.1)$$

and

$$\lambda_g = y_g - \eta_g \quad (2.2)$$

The tire contacts the ground in a finite area having a length which is designated as $2h$. The width of this area is assumed to be negligible small, that is, the ground contact area is assumed to be reduced to a ground contact line. The foremost ground contact point (in the direction of motion) is designated by the subscript 1, the rearmost point by the subscript 2 and the center point by the subscript 0. Except for braking and accelerating effects, the center point 0 has approximately the same horizontal X coordinate as that of the wheel axle.

Distances about the tire equator or circumference are measured in terms of the circumferential coordinate s whose origin is taken at the point O .

The wheel is assumed to move at constant velocity v approximately in the direction of the X -axis. The wheel is laterally inclined with respect to a vertical Z -axis by the tilt angle γ and is swiveled with respect to the XZ -plane by the swivel angle θ . Both tilt and swivel angles are assumed to be small, that is, $\cos \theta \approx \cos \gamma \approx 1$, $\sin \gamma \approx \gamma$ and $\sin \theta \approx \theta$.

The center point of the wheel axle is located vertically at a distance r_3 from the XY ground plane and is laterally located with respect to the ground contact intersection of the wheel plane by a distance $r_3\gamma$ or with respect to the XZ -plane by η_3 where

$$\eta_3 = \eta_0 - r_3 \quad (2.3)$$

Tire Distortion

In this section a short discussion is given of the features of tire distortion which are pertinent to the derivative of the basic kinematic relations of this paper.

Experimental and theoretical considerations, such as have been given by Schlippe and Dietrich²⁸ and

²⁸ B. von Schlippe and R. Dietrich, Ibid.

Rotta,²⁹ respectively, indicate that if the tire equator in the ground contact region of a tire is subjected to arbitrary lateral distortion, then the lateral distortion of the tire equator off the ground λ_1 tends to die out as an exponentially decaying function of the circumferential displacement s (for example, see Figure 2(a)). Thus near tire point 1 off the ground the tire distortion tends to approach the pattern described by the equation

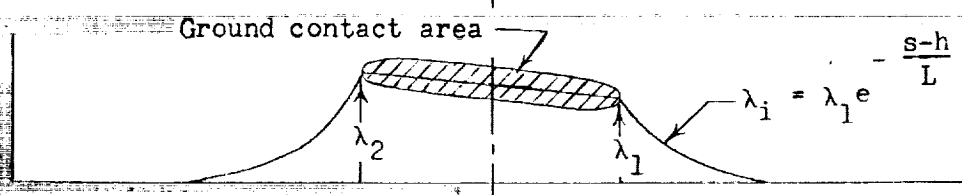
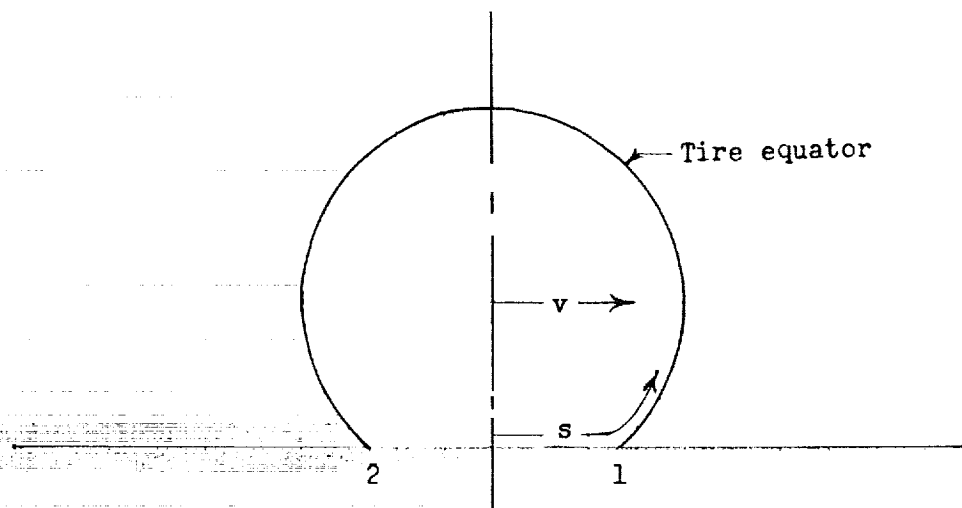
$$\lambda_1 = \lambda_{1e}^{-\frac{s-h}{L}} \quad (2.4)$$

and a similar equation applies near tire point 2. The exponential constant L is a tire characteristic having the dimension of a length and is called the relaxation length. It is noted that the value of relaxation length near point 2 is not necessarily exactly the same as that near point 1; however, since the former relaxation length is not used in this paper in any critical calculations, there is no point in taking into account the difference.

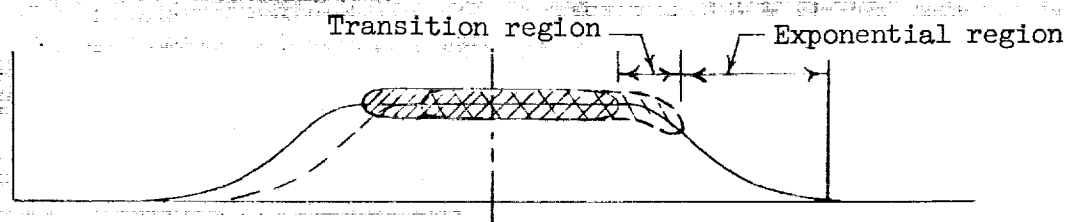
In regard to the accuracy of equation (2.4) very near to point 1, it should be emphasized that this exponential variation is only an expression of the equilibrium condition

²⁹

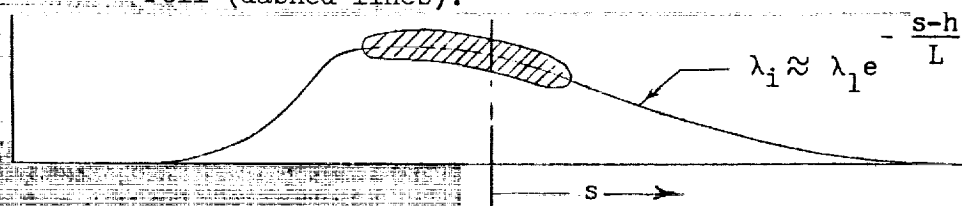
J. Rotta, op. cit.



(a) Assumed theoretical shape of tire equator distortion for a stationary twisted tire ($v = 0$).



(b) Actual shape of tire equator distortion for an untwisted tire at rest (solid lines) and just after starting to roll (dashed lines).



(c) Actual shape of tire equator distortion for a rolling tire.

FIGURE 2

TIRE EQUATOR DISTORTION

which the tire equator distortion would reach in the absence of any restraints. In actuality, it is obvious that there exist conditions where this distortion curve cannot be completely exponential in form. For example, for the case of pure lateral deflection of a stationary tire, the tire equator in the ground contact zone is (neglecting skidding) a straight line parallel to the wheel center plane and extending from point 1 to point 2. (See solid line outline in Figure 2(b).) Consequently, the existence of an exponential curve just to the right of point 1, and including point 1, would imply the existence of a sharp bend in the tire at point 1 such as is indicated in Figure 2(a). However, since a sharp bend is impossible because of finite tire stiffness, it follows that on a stationary tire in general the exponential variation given by equation (2.4) cannot be valid very close to point 1. However, experimental evidence does indicate that beyond a short transition region ahead of point 1 the tire equator distortion curve does have an essentially exponential character (see solid line outline in Figure 2(b)). As the wheel rolls ahead the nonexponential transition region of the tire equator is laid down or developed on the ground as it passes into the ground contact zone, and the more nearly exponential part of the equator curve moves down toward the ground (see dashed line outline in Figure 2(b)).

and is eventually developed on the ground, so that after rolling a short distance from rest and during normal rolling conditions (see Figure 2(c)) the tire equator distortion at the front end of the tire can approach the assumed exponential variation of equation (2.4).

At the rear end of the tire, the equator distortion curve during rolling does not so closely approximate an exponential variation since at the rear end there is no process of laying down or development such as is responsible for the exponential variation at the front end. However, since the rearward section of the tire equator is not used in any critical calculation in this paper, it is, nevertheless, assumed hereafter for simplicity that this equator curve is also exponential.

Now having given some reason for accepting equation (2.4) as the basic equation for tire equator lateral distortion near point 1 for rolling conditions, the total lateral displacement of the tire from the XZ-plane in this region can then from equation (2.1) be written in the form

$$y_1 = \eta_1 + \lambda_1 e^{-\frac{s-h}{L}} \quad (2.5)$$

and substituting the geometrical relation $\eta_1 = \eta_g - z\gamma$ (see Figure 1) gives

$$y_1 = \eta_g - z\gamma + \lambda_1 e^{-\frac{s-h}{L}} \quad (2.6)$$

This equation will not be utilized to establish some basic kinematic relations.

The Kinematic Equation

By making use of the physical concepts discussed earlier in this chapter together with equation (2.6), it is now possible to establish as follows the basic differential equation relating the tire deflection at the center of the ground contact area y_0 with the wheel coordinates η , θ and γ .

In the ground contact surface between tire and ground there is assumed to be perfect adhesion, that is, no skidding. As the tire rolls forward (arbitrarily swiveling, tilting and moving laterally) by a distance dx a new element of the tire of length ds above and in front of point 1 is laid down or developed on the ground. This tire element, before being laid down on the ground, had the lateral distortion variation given by equation (2.6). This equation,

after differentiation with respect to s , yields for a given instantaneous position of the tire the following rate of change of distortion

$$\frac{dy_1}{ds} = \frac{d\eta_g}{ds} - \gamma \frac{dz}{ds} - \frac{1}{L} \lambda_1 e^{-\frac{s-h}{L}} \quad (2.7)$$

and at the point 1 where $s = h$ and $y_1 = y_1$

$$\left(\frac{dy_1}{ds}\right)_1 = \left(\frac{d\eta_g}{ds}\right)_1 - \gamma \left(\frac{dz}{ds}\right)_1 - \frac{1}{L} \lambda_1 \quad (2.8)$$

Consider the term $\left(\frac{dz}{ds}\right)_1$. This is simply the sine of the angle between the ground and the tire equator at point 1 (see Figure 1). Just to the left of point 1 the tire is flattened on the ground or $\frac{dz}{ds} = 0$. If $\left(\frac{dz}{ds}\right)_1$ were different from zero this would then imply a sharp bend in the tire at point 1. However, because of the finite bending stiffness of an actual tire a sharp bend is impossible; thus $\left(\frac{dz}{ds}\right)_1 = 0$ and equation (2.8) reduces to the relation

$$\left(\frac{dy_1}{ds}\right)_1 = \left(\frac{d\eta_g}{ds}\right)_1 - \frac{1}{L} \lambda_1 \quad (2.9)$$

Further, since $\left(\frac{dz}{ds}\right)_1 = 0$, s is a horizontal coordinate near point 1. The rate of change of wheel lateral displacement η_g with respect to the horizontal coordinate at any given instant is just the swivel angle θ , hence it follows that

$$\left(\frac{dy_1}{ds}\right)_1 = \theta - \frac{1}{L} \lambda_1 \quad (2.10)$$

If the tire is assumed to have no sharp bend at point 1 $(dy_1/ds)_1 = (dy_g/ds)_1$ at this point. Then, since $(dy_g/ds)_1$ is the slope of the tire equator on the ground at point 1 and since no slipping is assumed to exist, it follows that this slope must coincide with the track of the rolling tire on the ground which is dy_1/dx . Thus

$$\frac{dy_1}{dx} = \theta - \frac{1}{L} \lambda_1$$

or designating differentiation with respect to x by the operator $D = \frac{d}{dx}$ and rearranging

$$LDy_1 = L\theta - \lambda_1 \quad (2.11)$$

$$\begin{aligned}
\eta_0 + l_1 \theta - \frac{Lh}{r} \gamma &= \eta_3 + l_1 \theta + \left(r_3 - \frac{Lh}{r} \right) \gamma \\
&= (1 + l_1 D + l_2 D^2 + \dots) y_0 \\
&= \left(1 + \sum_{n=1}^{\infty} l_n D^n \right) y_0 \\
&= (1 + LD) e^{hD} y_0 \\
&= (1 + Lv^{-1} D_t) e^{hv^{-1} D_t} y_0 \quad (2.20)
\end{aligned}$$

where

$$l_1 = L + h$$

$$l_2 = (2L + h)h/2$$

$$l_n = (nL + h)h^{n-1}/n!$$

Equation (2.20) furnishes several alternate forms of the basic kinematic equation (2.16) which are useful in later chapters.

CHAPTER III

FORCES AND MOMENTS ON THE WHEEL

In this chapter the primary forces and moments acting on a rolling wheel are discussed and where possible equations are set down for these quantities. These equations are utilized in Chapter III together with the preceding kinematic equation to establish the equations of motion for a rolling wheel.

Throughout this discussion all forces along the coordinate axes are considered positive if they tend to move the wheel in the direction of the positive coordinate axes; all moments about the coordinate axes X , Y or Z or other parallel axes are considered positive if they tend to produce wheel rotation from the positive Y -axis toward the positive Z -axis, from the positive Z -axis toward the positive X -axis or from the positive X -axis toward the positive Y -axis, respectively.

Lateral Elastic Force

First consider the lateral elasticity properties of a tire. If a static untilted tire is laterally deflected at its base with respect to its rim by a lateral $F_y \lambda$ it produces an equal spring reaction force roughly proportional to the mean lateral distortion λ_{mean} or, inversely, a lateral

tire distortion λ_{mean} creates a proportional ground force $F_{y\lambda}$. If the lateral distortion of the center of the ground contact line λ_0 is taken as the mean distortion then the elastic ground force is

$$F_{y\lambda} = K_{\lambda} \lambda_0 = K_{\lambda} (y_0 - \eta_0) = K_{\lambda} (y_0 - \eta_3 - r_3 \gamma) \quad (3.1)$$

where K_{λ} is the tire lateral spring constant or side stiffness. This relation is used in most papers. However, Schlippe and Dietrich³³ Rotta³⁴ do use a slightly different expression. These investigators take the tire mean lateral distortion equal to the average of the distortion at the leading and trailing edge points of the ground contact area (points 1 and 2) and thus obtain the equation

$$F_{y\lambda} = \frac{1}{2} K (\lambda_1 + \lambda_2) \quad (3.2)$$

instead of equation (3.1). The true equation for $F_{y\lambda}$ is probably more complicated than either of these two equations; however, since no plausible means of obtaining a better equation is available it appears advisable to select one of the above equations for use in this paper. Although it is

³³ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," op. cit.

³⁴ J. Rotta, op. cit.

possible that equation (3.2) may be slightly the better equation for a few special cases of wheel motion, equation (3.1) is much simpler to work with and in the majority of cases of wheel motion it makes little difference which of the two equations is used. Therefore, for the sake of simplicity, equation (3.1) is adopted hereafter in the analysis of this paper as the basic equation for the lateral force on a wheel due to tire lateral deformation.

Torsional Elastic Moment

Consider next the torsional elasticity properties of a tire. If a tire is twisted on the ground about a vertical axis by an angle α there arises a restoring ground moment roughly linearly proportional to the twist or

$$M_{za} = K_a \alpha \quad (3.3)$$

The tire twist is equal to the mean angle between the track of the tire on the ground and the wheel plane; that is, $\alpha = Dy_{\text{mean}} - \theta$. Taking the mean value of Dy_{mean} as Dy_0 gives

$$\alpha = Dy_0 - \theta \quad (3.4)$$

so that

$$M_{za} = K_a(Dy_0 - \theta) = K_a(v^{-1}D_t y_0 - \theta) \quad (3.5)$$

Most investigators of tire motion have used this relation. However, Schlippe and Dietrich³⁵ and Rotta³⁶ instead take the mean angle equal to $(\lambda_1 - \lambda_2)/2h$ and thus obtain the moment equation

$$M_{za} = \frac{K_a}{2h} (\lambda_1 - \lambda_2) \quad (3.6)$$

which leads to relatively more complicated equations of motion than does equation (3.5). However, there is no strong reason for believing equation (3.6) to be a significant improvement over the simpler equation (3.5). Therefore, in the analysis of this paper, the simpler equation is used.

It is noted that Melzer³⁷ has used the less accurate relation that the moment due to tire twist is

$$M_{za} = -K_a \theta \quad (3.7)$$

³⁵ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," op. cit.

³⁶ J. Rotta, op. cit.

³⁷ M. Melzer, op. cit.

which implies the relation $\theta \gg Dy_0$ (see equation (3.5)) which is, however, not true in all practical cases. Consequently, in regard to this point Melzer's theory should be viewed with some caution.

Tilt Elastic Force

Rotta³⁸ has shown that if a tire is tilted from the vertical Z-axis by an angle γ without lateral distortion of the equator ($\lambda_0 = 0$) there arises a restoring ground lateral force approximately linearly proportional to the tilt angle or

$$F_{y\gamma} = -K_\gamma \gamma \quad (3.8)$$

where K_γ is the constant of proportionality. Most authors (excepting Rotta) have not considered the effects of this force term although they have considered other effects of the same order of magnitude.

Vertical Load Center of Pressure

Under some circumstances the vertical load F_z influences the wheel motion. To consider this influence, it is necessary to know the locations of the center of pressure

³⁸

J. Rotta, op. cit.

of this force. In the XZ-plane (Figure 1) this center of pressure lies approximately below the wheel axle in line with the point O. In the YZ-plane the center of pressure is shifted laterally from the intersection of wheel plane and ground η_0 as a result of lateral distortion λ_0 and tilt γ . As a first approximation this shift may be taken as linearly dependent on λ_0 and γ so that the lateral center of pressure distance c from the XZ-plane becomes

$$\begin{aligned} c &= \eta_0 + c_\lambda \lambda_0 - c_\gamma \gamma \\ &= c_\lambda y_0 + (1 - c_\lambda) \eta_0 - c_\gamma \gamma \\ &= c_\lambda y_0 + (1 - c_\lambda) \eta_3 + \left[(1 - c_\lambda) r_3 - c_\gamma \right] \gamma \quad (3.9) \end{aligned}$$

where c_λ and c_γ are constants. (The signs of the terms are chosen such that c_λ and c_γ are positive numbers.)

Gyroscopic Moment Due to Tire Distortion

Consider next how a gyroscopic moment can arise in the case of a rolling untilted wheel with lateral distortion of the tire at the ground (Figure 3). While the solid rim and axle parts of the wheel are untilted, the elastic tire, due to the lateral deformation, is, on the average, tilted with respect to the wheel center plane by an amount

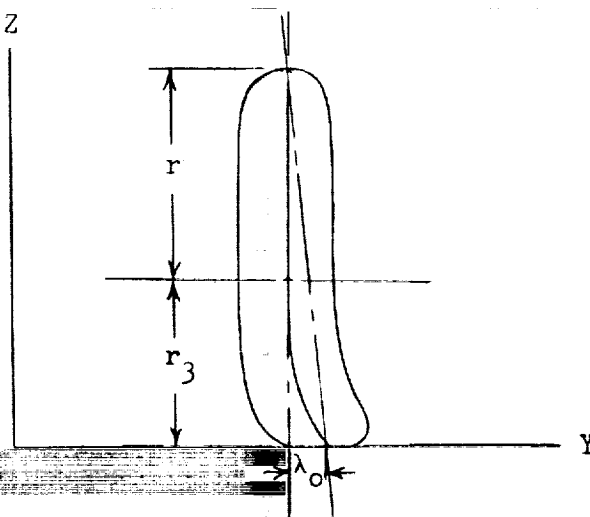


FIGURE 3

EFFECTIVE TIRE TILT DUE TO LATERAL DISTORTION OF TIRE

$\gamma_\lambda = \frac{\lambda_0 \tau_1}{r + r_z}$ where r is the tire radius and τ_1 is a

correction factor which indicates the effective fraction of the total tire mass which is tilted at this angle.

Kantrowitz,³⁹ who was apparently the only investigator to consider this at least theoretically interesting factor, has suggested that $\tau_1 \approx 1/2$. This tilting action produces an angular velocity $D_t \gamma_\lambda = \frac{D_t \lambda_0 \tau_1}{r + r_z}$ where D_t indicates differentiation with respect to time. This angular velocity together with the rotational velocity of the tire ω produces a gyroscopic moment about the Z-axis of magnitude

$$M_{z\lambda} = -I_{yt} \omega D_t \gamma_\lambda \quad (3.10)$$

where I_{yt} is the moment of inertia of the tire (excluding the solid rim and axle) about the wheel axle. By using equation (2.18) this equation can also be expressed in the form

$$M_{z\lambda} = -I_{yt} v^2 \frac{\omega}{v} D \gamma_\lambda \quad (3.11)$$

where the ratio v/ω is, to a good enough approximation for this secondary term, equal to the tire radius r .

³⁹ Arthur Kantrowitz, op. cit.

Then substituting for γ_λ and ω/v in equation (3.11) gives

$$M_{z\lambda} = - \frac{\tau_1 I_{yt} v^2}{r(r + r_3)} D\lambda_0$$

or abbreviating for later convenience and expressing the result in several alternate useful forms

$$M_{z\lambda} = -\tau v^2 D\lambda_0 = -\tau v^2 D(y_0 - \eta_0) = -\tau v D_t(y_0 - \eta_3 - r_3 \gamma) \quad (3.12)$$

where

$$\tau = \frac{\tau_1 I_{yt}}{r(r + r_3)} \quad (3.13)$$

Another method for deriving an expression for τ is discussed in a later section.

Gyroscopic Moment Due to Wheel Tilting

If the entire wheel structure tilts at an angular velocity $D_t \gamma$ then there arises another gyroscopic moment of magnitude

$$M_{z\gamma} = -I_{yw} \omega D_t \gamma \approx -I_{yw} v D_t \gamma / r \quad (3.14)$$

in addition to the term of equation (3.12). Here I_{yw} is the total polar moment of inertia of the wheel (including the tire) about its axle.

Gyroscopic Moment Due to Wheel Swiveling

If the wheel swivels at an angular velocity $D_t\theta$ then there also arises a tilting gyroscopic moment of magnitude

$$M_{x\theta} = -I_{yw}D_t\theta \approx -I_{yw}vD_t\theta/r \quad (3.15)$$

Tire Inertia Forces and Moments

This section is concerned with an examination of the influence of tire inertia forces and moments on a wheel rolling at high speeds. Two types of such inertia effects will be evaluated now in separate subsections. First, inertia forces and moments associated with lateral distortion and twisting of the tire will be evaluated and second, centrifugal forces and moments will be evaluated. Then the overall effects of these two inertia type forces and moments will be considered together in a separate subsection.

Inertia forces and moments due to lateral tire distortion.— At high rolling or shimmy velocities tire inertia forces and moments arise which are proportional to the

relative accelerations of the different parts of the tire (including the previously discussed gyroscopic moment due to tire lateral distortion). A rough estimate of these forces and moments can be made as follows. It is assumed that a fraction one-third of the total mass of the tire m_t is located on the periphery of the tire and is subjected to the same accelerations with respect to the wheel hub as are tire particles on the equator line, the remaining tire mass being assumed substantially undisturbed. The "active" mass of the tire per unit circumferential length is then $m_t/6\pi r$. The lateral acceleration of tire particles on the right hand side of the tire off the ground in Figure 1(a) will be considered first. The tire lateral distortion for this region is given by equation (2.4). The lateral relative velocity of a tire particle, obtained by differentiating this quantity with respect to time is

$$D_t \lambda_1 = (D_t \lambda_1 - \lambda_1 D_t s/L) e^{-\frac{s-h}{L}}$$

The quantity $D_t s$, which represents the peripheral velocity of tire particles with respect to the wheel axle is approximately equal to the negative of the rolling velocity v so

that the velocity expression becomes

$$D_t \lambda_1 = (D_t \lambda_1 + v \lambda_1 / L) e^{-\frac{s-h}{L}}$$

Differentiation of this result to give the tire particle relative acceleration yields the result

$$D_t^2 \lambda_1 = (D_t^2 \lambda_1 + 2v D_t \lambda_1 / L + v^2 \lambda_1 / L^2) e^{-\frac{s-h}{L}}$$

The corresponding inertia force for this part of the tire is obtained by integrating this acceleration times the active mass per unit length. This gives the force term

$$\frac{m_t}{6\pi r} \int_{s=h}^{\infty} s \pi r D_t^2 \lambda_1 ds \quad \text{and evaluation of this integral, after}$$

replacing the upper limit by infinity for simplification of the result (which introduces no significant error because of the rapidly decaying exponential function in $D_t^2 \lambda_1$), yields the result

$$\Delta F = - \frac{m_t L}{6\pi r} (D_t^2 \lambda_1 + 2v D_t \lambda_1 / L + v^2 \lambda_1 / L^2) \quad (3.16)$$

The corresponding inertia moment term ΔM is given by

$$- \frac{m_t}{6\pi r} \int_{s=h}^{\infty} s^2 \pi r r \sin \varphi D_t^2 \lambda_1 ds \quad \text{where } r \sin \varphi \text{ is the moment arm}$$

(see Figure 1) and s is related to φ by the relation $s - h = r(\varphi - \varphi_1)$ and $\varphi_1 = \sin^{-1} h/r$ so that the moment integral may be written in terms of φ in the form

$$\Delta M = \frac{m_t}{6\pi r} \int_{\varphi_1}^{\pi} r \sin \varphi (D_t^2 \lambda_1 + 2vD_t \lambda_1/L + v^2 \lambda_1/L^2) e^{-\frac{r(\varphi - \varphi_1)}{L}} r d\varphi$$

The evaluation of this integral, after replacing the upper limit by infinity (which introduces no appreciable error), then yields the expression

$$\Delta M = - \frac{m_t r L (h + L \sqrt{1 - h^2/r^2})}{6\pi(L^2 + r^2)} (D_t^2 \lambda_1 + 2vD_t \lambda_1/L + v^2 \lambda_1/L^2) \quad (3.17)$$

In a similar manner, for tire particles off the ground on the left hand side of the tire in Figure 1(a) the following expressions are obtained for the inertia force and moment.

$$\Delta F = - \frac{m_t L}{6\pi r} (D_t^2 \lambda_2 - 2vD_t \lambda_2/L + v^2 \lambda_2/L^2) \quad (3.18)$$

$$\Delta M = \frac{m_t r L (h + L \sqrt{1 - h^2/r^2})}{6\pi(L^2 + r^2)} (D_t^2 \lambda_2 - 2vD_t \lambda_2/L + v^2 \lambda_2/L^2) \quad (3.19)$$

In these two expressions it has been assumed, for reasons previously discussed, that the relaxation length L for both ends of the tire is the same.

To obtain the inertia forces and moments for tire particles in the ground contact area it is recognized that for practically all cases where inertia forces are important the ground contact line is close to a straight line so that the lateral distortion for tire particles in this region can be expressed fairly well by the equation

$$\lambda_g = \lambda_0 + sa$$

and the corresponding velocity and acceleration are

$$D_t \lambda_g = D_t \lambda_0 + s D_t a - va$$

$$D_t^2 \lambda_g = D_t^2 \lambda_0 + s D_t^2 a - 2v D_t a$$

The total inertia force for this region is then

$$\begin{aligned} \Delta F &= - \frac{m_t}{6\pi r} \int_{s=-h}^{s=h} D_t^2 \lambda_g ds \\ &= - \frac{m_t h}{3\pi r} (D_t^2 \lambda_0 - 2v D_t a) \end{aligned} \quad (3.20)$$

and the inertia moment is

$$\begin{aligned}\Delta M &= -\frac{m_t}{6\pi r} \int_{s=-h}^{s=h} s D_t^2 \lambda_g ds \\ &= -m_t h^3 D_t^2 a / 9\pi r\end{aligned}\quad (3.21)$$

The total inertia force F_{yi} obtained by summing up the force terms in equations (3.16), (3.18) and (3.20) can be stated conveniently in terms of λ_0 and a by using the relations $\lambda_1 + \lambda_2 = 2\lambda_0$ and $\lambda_1 - \lambda_2 = 2ha$ which are valid for a substantially straight ground contact line. This gives the result

$$F_{yi} = \frac{m_t}{3\pi r} (l_1 D_t^2 \lambda_0 + v^2 \lambda_0 / L) \quad (3.22)$$

where $l_1 = L + h$, and similarly for the total inertia moment M_{zi}

$$\begin{aligned}M_{zi} &= -\frac{m_t}{3\pi} \left[\left(\frac{rhL(h + L\sqrt{1 - h^2/r^2})}{L^2 + r^2} + \frac{h^3}{3r} \right) D_t^2 a + \right. \\ &\quad \left. \frac{r(h + L\sqrt{1 - h^2/r^2})}{L^2 + r^2} (2vD_t \lambda_0 + hv^2 a / L) \right] \quad (3.23)\end{aligned}$$

To partly evaluate the significance of these inertia expressions consider first the inertia force for sinusoidal oscillations such that $\lambda_0 = \lambda_{0m} \sin vt$ and therefore $D_t^2 \lambda_0 = -v^2 \lambda_0$ so that equation (3.22) may be restated as

$$F_{yi} = - \frac{mt}{3\pi r} (v^2/L - l_1 v^2) \lambda_0 \quad (3.24)$$

In order to interpret the significance of the inertia force, it is noted that the important tire force quantity which is of importance for the subsequent analysis is the net tire force F_{yn} acting on the wheel which is equal to the sum of the ground force $F_{y\lambda}$ and the inertia force F_{yi} or

$$F_{yn} = F_{y\lambda} + F_{yi} \quad (3.25)$$

Next, consider how the inertia force modifies the ground force $F_{y\lambda}$ which was previously set equal to $K_\lambda \lambda_0$ for the case of a static tire (see equation (3.1)). In the dynamic case, the relation between ground force and lateral tire distortion may be modified by the inertia effect. As a first approximation for this modification effect it is assumed hereafter that the modification of the ground force is proportional to the inertia force or

$$F_{y\lambda} = K_\lambda \lambda_0 - \eta_y F_{yi} \quad (3.26)$$

where η_y is a number whose absolute value will be less than unity if the modification of the ground force due to the inertia force is less than the inertia force itself.

After combining equations (3.24), (3.25) and (3.26) the following equation for the net tire force F_{yn} is obtained.

$$F_{yn} = \left[K_{\lambda} - (1 - \eta_y) \frac{m_t}{3\pi r} (v^2/L - l_1 v^2) \right] \lambda_0 \quad (3.27)$$

and from the form of this equation it can be seen that, insofar as the ratio of net tire force to lateral deformation is concerned, the effect of the inertia force can be considered equivalent to a change in tire lateral stiffness ΔK_{λ_1} equal to

$$\Delta K_{\lambda_1} = -(1 - \eta_y) \frac{m_t}{3\pi r} (v^2/L - l_1 v^2) \quad (3.28)$$

Similarly from the inertia moment equation (3.23), it can be concluded from an examination of the terms containing a that part of the effect of this inertia moment is to change the tire torsional stiffness by an amount ΔK_a where

$$\Delta K_a = - \frac{(1 - \eta_z) m_t}{3\pi} \left[\frac{r h (h + L \sqrt{1 - h^2/r^2})}{L^2 + r^2} (v^2/L - l_0 v^2) - h^3 v^2 / 3r \right] \quad (3.29)$$

and where η_z is a number for the torsional stiffness similar to η_y for the lateral stiffness. The remaining inertia moment term in equation (3.23) proportional to $D_t \lambda_0$, is simply the previously discussed gyroscopic moment due to lateral tire distortion. By comparing this term with the previously obtained equation (3.12), it is seen that the coefficient τ may be expressed by the equation

$$\tau = \frac{2m_t r (h + L \sqrt{1 - h^2/r^2})}{3\pi(L^2 + r^2)} \quad (3.30)$$

which usually gives approximately the same result as the previously derived equation (3.13) with Kantrowitz's assumption of $\tau_1 = 1/2$. In regard to the question of in what velocity range are the above tire stiffness changes important, it is convenient to postpone such a discussion until after a derivation of the effects of centrifugal forces has been made.

Effects of centrifugal forces.— Another inertia effect which may become significant at high speeds is produced by the centrifugal forces acting on the individual mass elements of the tire. The action of these centrifugal forces appears to be to increase the tire stiffness as will now be demonstrated by making use of a crude analysis which

should give a qualitative idea of the size of this effect, but which should not be regarded as possessing any strong quantitative merit.

For the purpose of this estimate, one-half of the mass of the tire is assumed to be concentrated in the tire sidewalls and the other half is assumed to be concentrated on the tire periphery.

If the tire lateral and torsional stiffnesses K_λ and K_α be assumed to be directly proportional to the tension in the tire sidewalls, then there will be two sources of tire stiffness, namely, inflation pressure, which produces a sidewall tension approximately equal to $w p$ per unit circumferential distance, (where w is the tire width) and centrifugal force, which produces the sidewall tension $\frac{1}{2} \cdot \frac{m_t}{2\pi r} \cdot \frac{v^2}{r}$ corresponding to the peripheral tire mass $1/2 m_t$. Thus the tire lateral stiffness may be expressed in the form

$$K_\lambda \sim 4\pi r^2 w p + m_t v^2$$

or equivalently as

$$\begin{aligned}
 K_{\lambda} &= K_{\lambda \text{ static}} \left(1 + \frac{m_t v^2}{4\pi r^2 w_p} \right) \\
 &= K_{\lambda \text{ static}} + \frac{m_t v^2 K_{\lambda \text{ static}}}{4\pi r^2 w_p}
 \end{aligned}$$

It appears from this equation that the influence of centrifugal force is to increase the tire lateral stiffness by an amount $\Delta K_{\lambda j}$ where

$$\Delta K_{\lambda j} = \frac{m_t v^2 K_{\lambda \text{ static}}}{4\pi r^2 w_p} \quad (3.31)$$

and similarly for the torsional stiffness

$$\Delta K_{\alpha j} = \frac{m_t v^2 K_{\alpha}}{4\pi r^2 w_p} \quad (3.32)$$

Significance of tire inertia effects with respect to tire stiffness.- The significance of the ^{above} two just discussed tire inertia effects on the tire stiffness will now be considered.

First, for the lateral stiffness, the effective change K_{λ} from its static value ΔK_{λ} is obtained by adding the two

increments according to equations (3.28) and (3.31), which gives the following equation for the effective overall change in tire lateral stiffness as a function of rolling speed and shimmy frequency

$$\Delta K_{\lambda} = \frac{(1 - \eta_y)m_t l_1 v^2}{3\pi r} - \frac{(1 - \eta_y)m_t v^2}{3\pi r L} + \frac{m_t v^2 K_{\lambda}}{4\pi r^2 \omega_p} \quad (3.33)$$

The first term involving the shimmy frequency appears to be small enough in comparison with K_{λ} such that it can probably be neglected for most practical conditions. The last two terms have opposite signs if $\eta_y < 1$ and thus may represent two partly counterbalancing effects. The second term arose from the previous considerations of the lateral acceleration of tire particles and is seen to effectively tend to reduce tire lateral stiffness with increasing rolling velocity if $\eta_y < 1$. The last term arose from the previous considerations of centrifugal forces and is seen to effectively tend to increase tire lateral stiffness. These last two terms indicate that at high rolling speeds, if $\eta_y < 1$, the tire stiffness may either drastically decrease or increase, depending on which of the two terms is larger. However, both of these terms happen to be of the same order of magnitude and the derivations of both terms were based on too

crude concepts to justify conclusions regarding which term is larger. Thus, the only conclusion that can be drawn is that at sufficiently high rolling speeds, drastic changes in tire lateral stiffness may occur. Whether the stiffness increases or decreases can probably be settled only by experiment.

To give some quantitative measure of the velocity at which these inertia effects become of significance, some calculations were made to determine the velocity at which the magnitude of the second term in equation (3.33) becomes equal to K_λ . By making use of Horne's static tire data for several modern aircraft tires,⁴⁰ it was found that this velocity averaged approximately $400 \sqrt{r}$ fps $\approx 270 \sqrt{r}$ mph where r is expressed in feet. Similar estimates for the velocity at which the third term in equation (3.33) becomes equal to K_λ indicated approximately this same velocity. Moreover, since this velocity is relatively high compared with normal present day landing speeds, it appears that the inertia effects on tire lateral stiffness considered here can probably, as a rule, be neglected.

For the torsional stiffness of a tire, the overall effective change in torsional stiffness ΔK_a due to tire

⁴⁰ Walter B. Horne, "Static Force-Deflection Characteristics of Six Aircraft Tires Under Combined Loading," NACA TN 2926, 1953, 92 pp.

inertia and centrifugal forces is obtained by adding the two increments according to equations (3.29) and (3.32), which gives the equation

$$\Delta K_a = - \frac{(1 - \eta_z)m_t}{3\pi} \left[\frac{rh(h + L \sqrt{1 - h^2/r^2})}{L^2 + r^2} (v^2/L - Lv^2) - \right. \\ \left. h^3 v^2 / 3r \right] + \frac{m_t v^2 K_{a_static}}{4\pi r^2 w_p} \quad (3.34)$$

This equation is parallel to equation (3.33) for the lateral stiffness so that statements made previously concerning the lateral stiffness apply here also.

Other inertia effects.— The preceding discussion of inertia effects suggests that one effect of tire inertia is to change tire stiffness at high speeds and to introduce a gyroscopic moment. However, it should be recognized that there are other inertia effects which will come into play probably at velocities close to those where the previously mentioned inertia effects arise. For example, the basic kinematic equation is based on the assumption of an exponentially distorted tire equator line corresponding to a definite "static" relaxation length. This assumption can be safely assumed to be valid (if it is valid at all) only for

conditions where the elastic forces in the tire predominate over the inertia forces. Where inertia forces are strong in comparison with elastic forces, it is at least doubtful whether the relaxation length remains constant.

While there are undoubtedly other effects of tire inertia in addition to the ones just discussed, it appears probable that the importance of many of these effects, discussed or not discussed, can be assessed by means of the following summary statement. The major effects of tire inertia on the rolling motion appear to come into play at a velocity of order of magnitude $400 \sqrt{r}$ fps $\approx 270 \sqrt{r}$ mph where r is expressed in feet. For velocities considerably smaller than this velocity, most inertia effects can probably be safely neglected; for velocities of this order of magnitude or higher, it is possible that many of the basic assumptions of this paper, and of most other papers on this subject, may be subject to considerable error.

Hysteresis Forces and Moments

In addition to the forces and moments just discussed, there are also certain damping forces and moments which arise as a consequence of the sometimes considerable hysteresis losses which arise in the distortion of elastic tires. It appears probable that these hysteresis effects are only

important at low rolling speeds (more specifically for low ratios of v/v_r).⁴¹ However, for simple types of landing gears attached to a rigid airplane, the low speed rolling condition could be the critical condition for design purposes; thus it is questionable whether the hysteresis effects can be neglected for accurate design calculations. On the other hand, neglect of the hysteresis effects would usually lead to conservative results but still the degree of conservatism might be excessive. } *awk*

Apparently the only significant attempt to deal with this hysteresis problem has been given by Schlippe and Dietrich, who have presented some plausible assumptions for dealing with hysteresis effects, but have not attempted to solve the relatively complex problem of exploiting these assumptions in detail.⁴²

A treatment of this hysteresis problem is considered beyond the scope of the present investigation.

Structural Forces and Moments

The preceding discussion covers the major ground forces and moments and the gyroscopic moments acting on the

⁴¹ H. Fromm, op. cit.

⁴² B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

wheel. In addition to these forces and moments, there exist the forces and moments acting on the wheel from the supporting structure. These will be designated as F_{ys} for the net structural force parallel to the Y-axis, M_{xs} for the net structural lateral tilting moment and M_{zs} for the net structural swiveling moment. These forces and moments include shimmy damper moments, spring restoring moments, inertia forces in a landing gear structure (exclusive of the wheel inertia force) and spring forces arising from the flexibility of a landing gear strut or of the fuselage of an airplane. In general, the majority of these forces and moments can probably be considered to be approximately linear in behavior except for shimmy damper moments, however, even these moments can be replaced as a first approximation by equivalent linear damping moments by using the method developed by Jacobsen.⁴³

Within the scope of a linear theory, these structural forces and moments will depend in a linear manner on the wheel center coordinates η_3 , θ and γ according to expressions of the type

$$F_{ys} = T_1(D_t)\eta_3 + T_2(D_t)\theta + T_3(D_t)\gamma \quad (3.35)$$

$$M_{xs} = T_4(D_t)\eta_3 + T_5(D_t)\theta + T_6(D_t)\gamma \quad (3.36)$$

⁴³ S. Timoshenko, Vibration Problems in Engineering. Second Edition; New York: D. Van Nostrand Company, Inc., 1937, Pp. 57-61.

$$M_{zs} = T_7(D_t)\eta_3 + T_8(D_t)\theta + T_9(D_t)\gamma \quad (3.37)$$

where the T 's are functions of the differential operator D_t , sometimes called transfer functions, whose specific forms will depend on the type of landing gear in question.

CHAPTER IV

EQUATIONS OF MOTION

The equations of motion for a rolling wheel are derived and briefly discussed in this chapter.

Derivation of the Equations of Motion

To obtain the first equation of motion, the sum of the lateral forces acting on the wheel parallel to the Y-axis is set equal to the corresponding inertia reaction. This gives the equation

$$F_{ys} + K_{\lambda}(y_0 - \eta_3 - r_3\gamma) - K_{\gamma}\gamma = m_w D_t^2 \eta_3 \quad (4.1)$$

(see equations (3.1) and (3.8)), where the first term in equation (4.1) is the structural force, the second term is the net force on the wheel resulting from tire elastic and inertia forces ($K_{\lambda} = K_{\lambda_{static}} + \Delta K_{\lambda}$ where ΔK_{λ} is given by equation (3.33)), the third term is the lateral ground force resulting from tire tilt and m_w is the mass of the wheel (including the tire).

By setting the sum of the lateral tilting moments about the wheel center equal to the inertia reaction, the equation

$$M_{xs} + P \left\{ c_\lambda y_0 + (1 - c_\lambda) \eta_3 + [(1 - c_\lambda) r_3 - c_\gamma] \gamma - \eta_3 \right\} + \\ \left[K_\lambda (y_0 - \eta_3 - r_3 \gamma) - K_\gamma \gamma \right] r_3 - I_{yw} v D_t \theta / r = I_{xw} D_t^2 \gamma \quad (4.2)$$

is obtained (see equations (3.1), (3.8), (3.9) and (3.15)), where the first term in equation (4.2) is the structural amount, the second term is the moment resulting from the vertical ground load, the third term is the moment of the ground forces resulting from tire lateral distortion and tilt and the fourth term is the gyroscopic moment resulting from the swiveling motion of the wheel and where I_{xw} is the moment of inertia of the wheel about an X-axis (or a Y-axis) through its center.

By setting the sum of the swiveling moments about the wheel center equal to zero, the equation

$$M_{zs} + K_a (v^{-1} D_t y_0 - \theta) - \tau v D_t (y_0 - \eta_3 - r_3 \gamma) - \\ I_{yw} v D_t \gamma / r = I_{xw} D_t^2 \theta \quad (4.3)$$

is obtained (see equations (3.5), (3.12) and (3.14)), where the first term in equation (4.3) is the structural moment, the second term is the net moment resulting from tire elastic and inertia forces exclusive of the gyroscopic moment due to tire lateral distortion ($K_a = K_{a_{static}} + \Delta K_a$) where ΔK_a

is given by equation (3.34), the third term is the gyroscopic moment resulting from tire lateral distortion, and the fourth term is the gyroscopic moment resulting from wheel lateral tilt.

Equations (4.1), (4.2), (4.3) and (2.16) or (2.20), together with the three auxiliary equations (3.35) to (3.37), are the basic equations of motion for the motion of an elastic wheel and if the T-functions in equations (3.35) to (3.37) are known for a particular landing gear, these equations can be solved simultaneously to determine the rolling behavior of the gear.

Next the question arises as to how to most profitably solve these equations for practical landing gear problems. There are essentially two methods of attack, either exact or approximate solution of the equations. In regard to exact solutions, it should be noted that in the past such solutions (omitting some of the less important previously mentioned terms) have been made only for the simplest case of a rigid swiveling landing gear attached to a rigid fuselage.^{44,45} While the exact solution of these equations

⁴⁴ B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

⁴⁵ J. Rotta, op. cit.

for more complex problems does not appear to present any insurmountable difficulties, it can, however, lead to the solution of relatively complex transcendental equations such that it is worth while to examine the possibility of finding simpler systematic approximations to the general equations.

A second reason for investigating systematic approximations to the summary theory arises in connection with the correlation of the summary theory with the other existing theories. Superficially, in its present form, the summary theory does not too closely resemble most of the other existing theories. However, by making use of the approximations which follow, it is fairly easy to see the correlations between the different theories.

The next chapter of this paper is concerned with the problem of establishing a series of systematic approximations to the general equations and the chapter following that one deals with the correlation of these approximations with the other existing theories of wheel motion. To expedite some of the discussion in these later chapters, it is convenient to digress slightly here to consider one special exact solution of the general equations, namely, for the case of steady yawed rolling.

Steady Yawed Rolling

Consider the case of an untilted wheel which rolls at constant velocity at a constant small swivel or yaw angle. For this special case, $y_0(x + h) = y_0(x) = \text{constant}$, $\theta = \text{constant}$ and $\eta_3 = \gamma = 0$ so that equations (2.2) (with y_0 for y_g), (2.16), (4.1) and (4.3), respectively, reduce to the relations

$$\lambda_0 = y_0 \quad (4.4)$$

$$y_0 = (L + h)\theta = l_1\theta \quad (4.5)$$

$$F_{ys} + K_\lambda y_0 = 0 \quad (4.6)$$

$$M_{zs} - K_\alpha \theta = 0 \quad (4.7)$$

By combination of equations (4.4) and (4.5) the tire lateral distortion is found to be

$$\lambda_0 = l_1\theta \quad (4.8)$$

By combination of equations (4.5) and (4.6) the lateral force on the wheel is found to be

$$F_{ys} = -l_1 K_\lambda \theta$$

The quantity $l_1 K_\lambda$, which represents the lateral force per unit yaw angle, is an important tire characteristic which is called the cornering power or lateral guiding characteristic of the tire. Later in this paper, it is found convenient to represent this quantity by a single symbol N where

$$N = l_1 K_\lambda \quad (4.9)$$

Another property of the steady yawed rolling condition which is of some interest is the distance of the center of pressure of the lateral force behind the center of the tire, which is sometimes called the pneumatic castor $\epsilon = -M_{zs}/F_{ys}$. This quantity, according to equations (4.5) to (4.9), is equal to

$$\epsilon = -M_{zs}/F_{ys} = K_a/N \quad (4.10)$$

CHAPTER V

SYSTEMATIC APPROXIMATIONS TO THE SUMMARY THEORY

In this chapter, a discussion is given of the possibilities for simplifying the preceding equations of motion and a series of systematic approximations to the general equations are set down.

First it is noted that all but one of the equations of motion (equations (3.35) to (3.37), (4.1), (4.2) and (4.3)) are usually simple linear equations and present no great difficulties. The exception is the kinematic equation which was originally transcendental in form (equation (2.16)) and was later expressed in the form of an infinite series of linear terms (equation (2.20)). The most promising way to simplify this equation appeared to be to assume that the series expansion in the infinite series form of the kinematic equation (equation (2.20)) is a rapidly convergent series such that all terms in the series above a certain value of n can be neglected. The question as to what is the rapidity of convergence of the series and its significance cannot be fully answered without a knowledge of the particular landing gear configuration considered. However, some insight into this question can be obtained by considering the case of purely sinusoidal oscillations of the form $y_0 = e^{i v_1 x}$ where the quantity v_1 will be called the path

frequency. Substitution of this expression into the infinite series in y_0 in equation (2.20) yields the result

$$\left(1 + \sum_{n=1}^{\infty} l_n D^n\right) y_0 = (p_{1\infty} + ip_{2\infty}) y_0 \quad (5.1)$$

where

$$\left. \begin{aligned} p_{1\infty} &= 1 - l_2 v_1^2 + l_4 v_1^4 - \dots \\ p_{2\infty} &= l_1 v_1 - l_3 v_1^3 + l_5 v_1^5 - \dots \end{aligned} \right\} (5.2a)$$

Another form for the p 's can be obtained by substituting the relation $y_0 = h_1 x$ into equation (2.16). This gives the results

$$\left. \begin{aligned} p_{1\infty} &= \cos v_1 h - L v_1 \sin v_1 h \\ p_{2\infty} &= \sin v_1 h + L v_1 \cos v_1 h \end{aligned} \right\} (5.2b)$$

The rate of convergence of the p -series of equation (5.2a) can be tested for any given frequency by substituting numerical values of L , h and v_1 into equations (5.2a) and (5.2b) and comparing the individual terms. A typical comparison is shown in Figure 4 for the conditions $L = 0.8r$, and $h = 0.5r$. The abscissa of this plot represents the

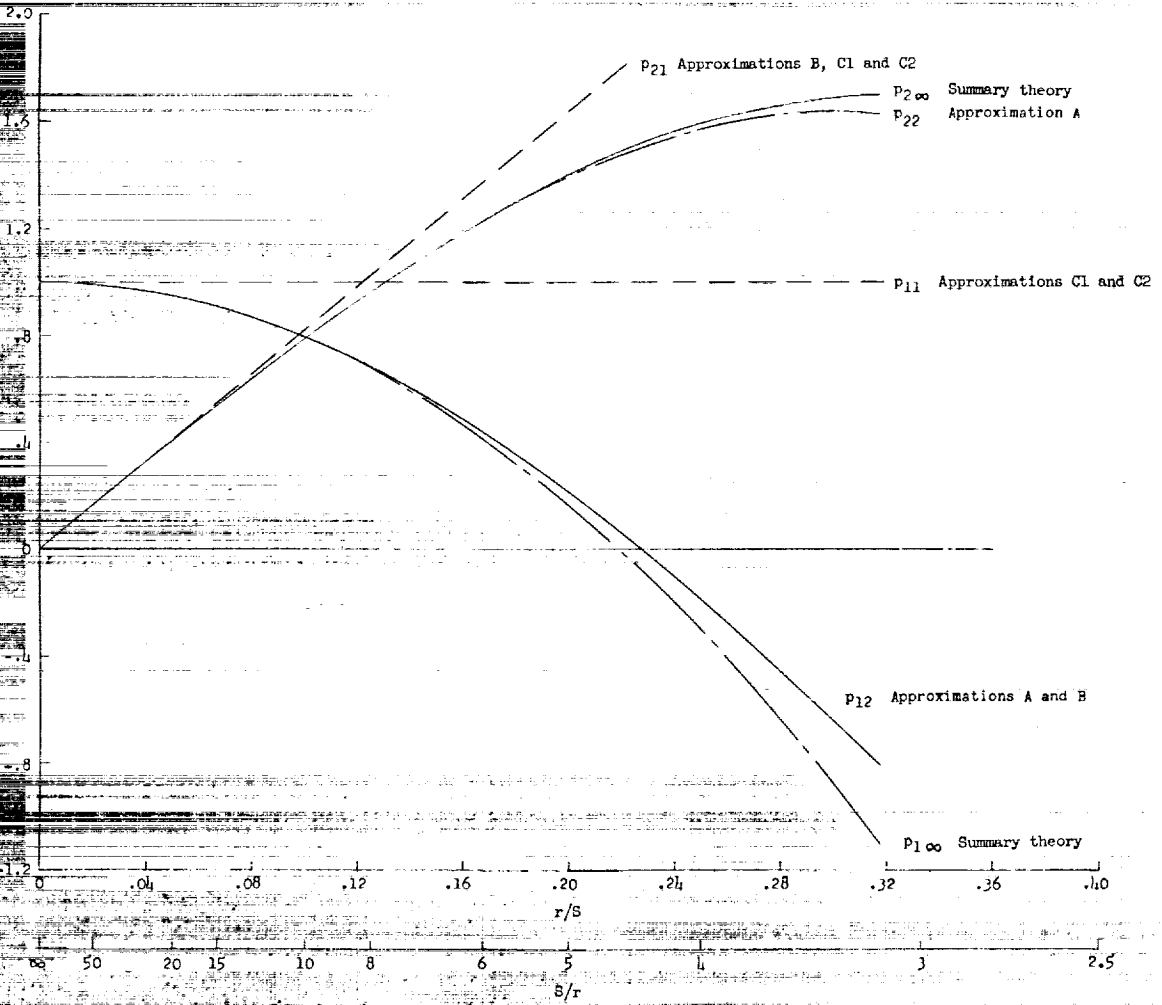


FIGURE 4

VARIATION OF p_1 AND p_2 WITH SHIMMY WAVE LENGTH

oscillation's wave length $S = 2\pi/v_1$ and the ordinate represents the p functions. The term labeled p_{12} means that this curve represents the sum of the first two terms in the $p_{1..}$ series and similarly for the other terms. (The approximation letters are explained later.) From this Figure it is seen that the series converge very rapidly. From a purely qualitative point of view, the Figure might be considered to indicate that for dealing with shimmy wave lengths greater than approximately four tire radii, the use of two terms in each series fairly well represents the exact variations, for wave lengths greater than approximately 6 radii, one term in the p_2 series and two in the p_1 series are sufficient and for wave lengths greater than about 20 radii, one term in each series is sufficient. (The numerical values of wave length cited here, of course, only represent order of magnitude and are not necessarily directly quantitatively significant.) To correlate these observations with the conditions of wave length likely to be encountered in practice, it can be stated that the experimental data of Schlippe and Dietrich^{46,47} and Kantrowitz,⁴⁸ which are

⁴⁶ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," op. cit.

⁴⁷ B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

⁴⁸ Arthur Kantrowitz, op. cit.

probably fairly typical in this respect, demonstrate wave lengths which are about 4 radii long at zero rolling velocity and which increase with increasing rolling velocity. Thus it appears possible that the use of only a few terms in the series expansion may lead to a reasonable prediction of shimmy characteristics for practical operating conditions.

With the preceeding considerations in mind, the following approximations to the general wheel motion equations of the summary theory were established.

Approximation A

As a first approximation for the general kinematic equation (2.20), all terms for $n > 3$ are neglected. This gives the approximate differential equation

$$y_0 + l_1 D y_0 + l_2 D^2 y_0 + l_3 D^3 y_0 = \eta_0 + l_1 \theta - \frac{l_1 h}{r} \gamma \quad (5.3)$$

This equation, together with all of the general force and moment equations previously discussed, is referred to hereafter as approximation A.

Approximation B

A second less exact approximation for equation (2.20) is obtained by letting $l_n = 0$ for $n > 2$. Thus

$$y_0 + l_1 D y_0 + l_2 D^2 y_0 = \eta_0 + l_1 \theta - \frac{l L h}{r} \gamma \quad (5.4)$$

This equation will be referred to as approximation B.

Approximation C1

Another cruder approximation for the general differential equation (2.20) is obtained by neglecting all terms in the series for $n > 1$. This gives the differential equation

$$y_0 + l_1 D y_0 = \eta_0 + l_1 \theta - \frac{l L h}{r} \gamma \quad (5.5)$$

which will be referred to as approximation C1.

Approximation C2

As a slight simplification of approximation C1, the relatively unimportant, or at least questionable, term involving γ may be omitted in equation (5.5). This gives the equation

$$y_0 + l_1 D y_0 = \eta_0 + l_1 \theta \quad (5.6)$$

which will be referred to as approximation C2.

With the aid of equations (2.2) and (3.4), equation (5.6) can be written in the more easily interpreted form

$$\lambda_0 = -l_1 a \quad (5.7)$$

or, by using in addition equations (3.1), (3.3) and (4.9), as

$$\lambda_0 = -\frac{N}{K_\lambda} a = \frac{F_y \lambda}{K_\lambda} = -\frac{NM_z a}{K_\lambda K_a} \quad (5.8)$$

which shows that for this approximation, the lateral distortion of the tire is directly proportional to the angular distortion.

The physical meaning of this approximation can be obtained by considering that equation (5.8) can also be obtained by letting the ground contact semi-length h approach zero in the general differential equation (2.20) (as was previously noted by Rotta⁴⁹) since all terms in the series for $n > 1$ and the tilt term contain the multiplication factor h . Then equation (2.20) becomes

$$y_0 + LDy_0 = \eta_0 + L\theta \quad (5.9)$$

or with equations (2.2), (3.4) and (5.9)

$$\lambda = -L\alpha \quad (5.10)$$

⁴⁹ J. Rotta, op. cit.

Also equation (4.9) for the yawed rolling becomes

$$N = K_{\lambda} L \quad (5.11)$$

and the combination of equations (5.10) and (5.11) gives

$$\lambda = - \frac{N}{K_{\lambda}} a \quad (5.12)$$

which is the same as equation (5.8) for any given combination of N and K_{λ} . Thus essentially, when written in the form of equation (5.8), this approximation C2 formally corresponds to the assumption of $h = 0$.

In regard to the accuracy of results obtained from this approximation, it can be qualitatively stated that reliable results should be expected only when the neglected quantity h is small with respect to the characteristic length S of the rolling motion in question (for example, the wave length of a sinusoidal oscillation). Fortunately, this condition is at least sometimes satisfied for practical rolling conditions.

Approximation D1

Before considering the next approximation, it should be remembered that all of the terms neglected in the preceding approximations were multiplied by the tire semi-length

h, thus these approximations implied the assumption of progressively smaller and smaller tire ground contact length or progressively larger and larger wave length. In order to make further simplifications, it is necessary to make some simplifying assumption about the other tire properties. Three such assumptions are now made to further simplify the equations of approximation C2. For the first approximation, to be called approximation D1, the simplification

$$b_1 = 0 \quad (5.13a)$$

is adopted. Then it follows from equation (5.7) that for finite a

$$\lambda_0 = 0 \quad (5.13b)$$

which is the basic equation for this approximation. Thus for this approximation, the tire is free to twist but not to deflect laterally. This, therefore, also implies infinite lateral stiffness or

$$K_\lambda = \infty \quad (5.13c)$$

For the simplest form of wheel shimmy due to tire elasticity and not to structural elasticity (considered later)

approximation D1 does not provide accurate information. For wheel shimmy due largely to structural elasticity rather than tire elasticity, this approximation may be of some value; actually most existing theories corresponding to this approximation have been developed for the primary purpose of considering the influence of structural elasticity on wheel shimmy.

Approximation D2

As a second simplification of approximation C2, the assumption

$$l_1 = \infty \quad (5.14a)$$

could be adopted and the corresponding theory is designated as approximation D2. From equation (5.7), it is evident that this approximation implies for finite λ_0 that

$$a = 0 \quad (5.14b)$$

which in turn implies

$$\left. \begin{aligned} N &= \infty \\ K_a &= \epsilon N = \infty \end{aligned} \right\} (5.14c)$$

Thus for this approximation, the tire is considered as torsionally rigid but laterally flexible.

Approximation D3

A third simplification of approximation C2 can be obtained by keeping the quantity l_1 finite but considering the tire to have both infinite lateral stiffness and infinite torsional stiffness or

$$K_\lambda = K_\alpha = N = \infty \quad (5.15)$$

This approximation, which is designated as approximation D3, thus represents the case of a rigid tire and consequently also implies $\alpha = \lambda_0 = 0$.

The seven preceding approximations A to D3 now furnish a choice of seven simplified approximations based on the summary theory and it remains to determine which, if any, of these approximations is the simplest one which can be used for any particular tire motion problem. While it is not yet possible to give a completely satisfactory answer to this question, some insight into the answer can be gained by comparing the various approximations with the other existing, at least partly successful, tire motion theories which are mostly closely related to these approximations. Such a comparison is carried out in the next two chapters of this paper.

CHAPTER VI

CLASSIFICATION OF EXISTING THEORIES

It is the purpose of this chapter to briefly review and evaluate the major previously published theories of wheel motion and to correlate these theories with the preceding summary theory of this paper and its approximations wherever such a correlation is possible. To accomplish this aim, each of the major previously published theories is first considered individually in a separate subsection and afterward an abbreviated overall summary classification is presented in a tabular form.

Individual Review and Evaluation of Existing Theories

Schlippe-Dietrich theory.- The tire motion theory of Schlippe and Dietrich,^{50,51,52} of course, corresponds directly to the summary theory of this paper since the summary theory was taken directly from their theory with only minor modifications. These modifications cover a more detailed consideration of some of the influences of lateral tilt and of tire inertia forces and moments. It should be noted, however,

⁵⁰ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," op. cit.

⁵¹ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

⁵² B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

that the Schlippe-Dietrich theory is more advanced than the summary theory of this paper in that it partly takes into account the effects of the width of the ground contact area. This effect, as was previously noted, is, however, probably not of great practical importance.

Rotta theory.- Rotta's tire motion theory⁵³ corresponds to the summary theory of this paper since it is also based on the Schlippe-Dietrich theory and represents a slight extension of that theory to more adequately take into account most of the effects of tire tilt and the width of the ground contact area. No inertia forces due to tire lateral distortion or centrifugal forces are discussed.

Bourcier de Carbon advanced theory.- Bourcier de Carbon⁵⁴ has developed two closely related theories of tire motion which are similar to approximations B and C2. The first of these will be referred to as the Bourcier de Carbon advanced theory and the second as the Bourcier de Carbon elementary theory. The advanced theory will be discussed first.

Bourcier de Carbon's advanced theory uses 5 basic tire properties which are correlated with those of the present paper by the following relations

⁵³ J. Rotta, op. cit.

⁵⁴ Christian Bourcier de Carbon, op. cit.

$$\underline{D} = \frac{1}{N}$$

$$\underline{T} = \frac{1}{K_{\lambda}}$$

$$\underline{S} = \frac{1}{K_{\alpha}}$$

$$\underline{\epsilon} = \frac{K_{\alpha}}{N}$$

$$\underline{R} = \frac{l_1}{l_2 K_{\alpha}} = \frac{2(L + h)}{K_{\alpha} h (2L + h)}$$

(6.1)

which were obtained by comparing this theory with the corresponding approximation B. The symbols of Bourcier de Carbon are underlined and do not necessarily bear any relation to any other not underlined symbols in this paper designated by the same letters. While the first four symbols bear a simple relation to those of the present paper, the fifth symbol R bears a more complicated relation which is worth some detailed consideration.

Bourcier de Carbon defined the tire property R as follows. If an untilted wheel is rolled forward while exposed to a constant turning moment about a vertical axis and with no side force, it will move in a circular path of a definite radius; R is defined as the reciprocal of the product of the turning moment and the path radius.

Unfortunately, however, this constant moment circle-rolling experiment is not easily performed. Therefore, the above equation for \underline{R} , which expresses \underline{R} in terms of the more easily measured and more fundamental quantities L , h and K_a is of importance for the use of the Bourcier de Carbon advanced theory.

In regard to the subject of tilt, Bourcier de Carbon omits many of the details considered in this paper. For example, he implicitly assumed $K_Y = c_\lambda = c_Y = \xi = 0$ and the inclination angle of a landing gear κ to be small (taking $\cos \kappa \approx 1$). However, these omitted tilt terms may be as important as the terms considered; therefore, Bourcier de Carbon's considerations of tilt are incomplete.

For the benefit of readers of Bourcier de Carbon's paper,⁵⁵ it should be noted that there exist certain misconceptions in the parts of that paper which deal with comparisons between theory and experiment. In particular, it appears that some of the experimental data quoted by Bourcier de Carbon from a paper by Schlippe and Dietrich⁵⁶ is misquoted or misinterpreted. Consequently, Bourcier de Carbon's conclusion that these experimental data provide a

⁵⁵ Christian Bourcier de Carbon, op. cit.

⁵⁶ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

remarkable check of his theory is not completely justified; actually these experimental data only provide an indirect fair check of the theory.

Greidanus theory.-- Another theory similar to the present approximation B, except for the influence of tilt, is the theory of Greidanus.⁵⁷ Greidanus considers the influence of tilt in much greater detail than does Bourcier de Carbon. However, he also fails to consider the force term proportional to K_Y ; thus his results also do not fully describe the influence of tilt.

In addition, Greidanus's kinematic equation differs from equation (5.4) for approximation B in that he has introduced a slightly different term associated with tilting of the tire. In the present terminology, Greidanus's equation reads

$$y_0 + l_1 D y_0 + l_2 D^2 y_0 = \eta_0 + l_1 \theta - l_2 \frac{\gamma}{r} \quad (6.2)$$

It is seen that the difference of the two equations lies in the coefficient of γ . For approximation B (equation (5.4)), the coefficient is

$$\frac{2l_h}{r} \quad (\text{Approximation B}) \quad (6.3)$$

⁵⁷

J. H. Greidanus, op. cit.

and for Greidanus's equation (after substituting for l_2 from equation (2.20))

$$\frac{(L + h/2)h}{r} \quad (\text{Greidanus}) \quad (6.4)$$

If ϵ is set equal to $(L + h/2)/L$ then the two coefficients are identical; thus Greidanus's kinematic equation can be considered to be a special case of the corresponding equation of approximation B.

No further detailed discussion of Greidanus' theory is given in this paper for the reason that lack of a translation of Greidanus' paper prohibits a complete understanding of some parts of the paper.

Bourcier de Carbon elementary theory.⁵⁸ Bourcier de Carbon's elementary theory corresponds to approximation C2 of this paper except for the minor shortcomings which were discussed in connection with the Bourcier de Carbon advanced theory. The only difference in Bourcier de Carbon's two theories is that the coefficient R in the elementary theory is taken as infinity as compared with the value given by equation (6.1) for the advanced theory. This corresponds to the assumption $l_2 = 0$ which was previously made in passing from approximation B to approximation C2 (compare equations

⁵⁸

Christain Bourcier de Carbon, op. cit.

(5.4) and (5.5)). The physical significance of $R = \infty$ is obvious from equation (6.1). It means $h = 0$.

Melzer theory.— The Melzer theory for tire motion⁵⁹ is also similar to that of approximation C2 except for details of the tilting process. Melzer's kinematic equation is identical with that of approximation C2 and of Bourcier de Carbon's elementary theory. However, Melzer's theory differs in that it takes the moment due to tire twist as proportional to the swivel angle ($-\theta$) rather than the total tire twist angle ($Dy_0 - \theta$). Logically, this assumption would appear justified only if $y' \ll \theta$. This, however, is not true in general. In connection with this point, it is interesting to note that for the simplest case of wheel shimmy (see Chapter VII, Case I), the Melzer approximation leads to one of the same stability boundaries and to the same limiting high speed shimmy frequency as the more correct approximation including the term in Dy_0 . This restricted agreement, however, hardly justifies the use of Melzer's approximation since predictions of the two approximations differ with respect to calculations of the divergence of the shimmy oscillations and with respect to another stability boundary. Moreover, for simple problems, the Melzer approximation is not significantly easier to solve than the more correct form including the Dy_0 term.

⁵⁹ M. Melzer, op. cit.

Moreland advanced theory.- Moreland has proposed three versions of a tire motion theory.^{60,61} The most advanced version of these is governed by the equation

$$a + C_1 D_t a = -\lambda_0 / l_1 = -F_{y\lambda} / N \quad (6.5)$$

or

$$C_1 l_1 v D^2 y_0 + l_1 D y_0 + y_0 = C_1 l_1 v D \theta + (l_1 - a) \theta \quad (6.6)$$

where C_1 is a time lag constant. This theory corresponds to a generalization of approximation C2 (with pneumatic castor neglected, that is, $\epsilon = 0$) to the extent that for $C_1 = 0$ equation (6.6) is identical with the basic equation for approximation C2. However, with $C_1 \neq 0$ this theory is not directly compatible with the summary theory and its approximations.

Moreland uses the following type of reasoning to establish this equation. First, for the case of steady yawed rolling, it is known that a yaw angle α is developed as a consequence of the application of a lateral force $F_{y\lambda}$

⁶⁰ William J. Moreland, "Landing-Gear Vibration," op.cit.

⁶¹ William J. Moreland, "The Story of Shimmy", op. cit.

according to the relation

$$\alpha = -F_{y\lambda}/N \quad (6.7)$$

which is the basic equation for approximation C2. However, for the dynamic rolling case obviously this equilibrium yaw angle cannot be established immediately upon application of a given side force; rather, a finite amount of time will be required for the equilibrium yaw angle to develop. Moreland has attempted to take this finite time lag into account by modifying equation (6.7) to the new form of equation (6.5). In the latter equation, the constant C_1 is a measure of the time lag of the yaw angle behind the applied force $F_{y\lambda}$.

This time lag term introduced by Moreland does not correspond exactly to any of the terms in the summary theory and to this extent Moreland's advanced theory is apparently incompatible with the summary theory. However, a partial reconciliation of the two theories can be obtained by recognizing that Moreland did not consider in detail the tire inertia forces and moments due to tire distortion. A possible interpretation of the time lag term is that it may provide a simplified expression for these inertia effects. In particular, it is interesting to note that, as will be shown later, the gyroscopic moment due to tire distortion produces some effects similar to those produced by the time lag term.

In regard to the question of the relative merits of the introduction of the overall time lag term and of the detailed inertia effects, it appears that this question can be decided only on the basis of relative agreement with experimental data. However, apparently the only existing experimental data containing time constant information which is suitable for such a comparison is Moreland's data⁶² which has not yet been published in detail. Consequently this paper cannot present a quantitative evaluation of the relative merits of these two approaches.

Moreland intermediate theory.- As a simpler approximation for the advanced theory, Moreland has stated⁶³ that the influence of the time lag term in the kinematic equation for his advanced theory (equation (6.5)) can be approximated for the usual range of shimmy frequencies by using the simpler kinematic equation

$$40l_1\alpha = -\lambda_0 \quad (6.8)$$

Insomuch as approximation C2 has the kinematic equation

$$l_1\alpha = -\lambda_0 \quad (5.7)$$

⁶² Ibid.

⁶³ William J. Moreland, "Landing-Gear Vibration," op. cit.

and approximation D2 has the kinematic equation (5.14b), which could be written in the form

$$-a \approx -\lambda_0$$

it then follows from a comparison of these last three equations that Moreland's intermediate theory is a theory which falls somewhere between approximations C2 and D2. Since Moreland has not offered any concrete justification for this approximation, it does not appear warranted to discuss it in further detail in this paper.

Moreland elementary theory.- Moreland's most elementary theory corresponds directly to approximation D3, the case of a completely rigid tire, except that it, like Moreland's other two theories, does not take into account the pneumatic castor ($\epsilon = 0$).

Temple elementary theory.- Temple has proposed an elementary theory for the motion of tires which is identical with approximation D1.⁶⁴ Temple has chosen the most general form of this approximation in that he has considered both the tire torsional stiffness K_a (indirectly interpreted as an increase in trail) and the cornering power N .

In regard to the general applicability of Temple's elementary theory, it should be noted that this theory was

⁶⁴ G. Temple, RAE Report No. AD 3148, op. cit.

developed before there was available experimental evidence pointing to the need for more detailed considerations of tire lateral stiffness. Subsequently, Temple has indicated a need for more refined considerations of the tire⁶⁵ and has developed independently a theory similar to the theory of Schlippe and Dietrich. (This theory is as yet unpublished but has been partly discussed by Hadekel.)⁶⁶

Maier theory.- Maier has proposed a simplified theory similar to approximation D1 with the difference that he makes the added assumption that the tire torsional stiffness K_a is zero.⁶⁷ In regard to this theory, like that of Temple, it should be noted that the theory was developed before there existed much experimental evidence pointing to the need for more refined considerations for shimmy behavior.

Taylor theory.- Taylor, in a brief paper,⁶⁸ suggested another tire motion theory which corresponds to approximation D2 except that details of the tilt process are omitted.

⁶⁵ G. Temple, "Note on American Work on Kinematic and Dynamic Shimmy," RAE Report No. AD 4056, 1940, 9 pp.

⁶⁶ R. Hadekel, "The Mechanical Characteristics of Pneumatic Tires," S. and T. Memo. No. 5/50, British Ministry of Supply, 1950, 146 pp.

⁶⁷ E. Maier, op. cit.

⁶⁸ J. Lockwood Taylor, op. cit.

Kantrowitz and Wylie theories.- The preceding theories for tire motion, which seem to cover most of the known theories, may all be considered as closely related to the summary theory of this paper. However, there exist two other well known theories by Kantrowitz⁶⁹ and Wylie⁷⁰ which apparently cannot be derived from the summary theory and thus cannot be accurately classified here with respect to the other theories. The best that can be said for their classification is that they possess some of the merits of approximation B but in other respects are inferior to approximation D1. To point out the deficiencies of these two theories, it is sufficient to consider two simple cases of tire motion as follows.

The first case to be considered is the steady straight line motion of a nonswiveling rolling wheel which is not yawed with respect to its direction of motion and which has no lateral forces or moments acting on the wheel. Obviously for this case, there will be no lateral distortion of the tire or

$$\lambda_0 = 0$$

⁶⁹ Arthur Kantrowitz, op. cit.

⁷⁰ Jean Wylie, op. cit.

On the other hand, Kantrowitz's basic kinematic equation, which is

$$\lambda_0 + LD\lambda_0 = L\theta - l_2 D\theta \quad (6.9)$$

gives for this steady unyawed case (with $D\lambda_0 = D\theta = 0$)

$$\lambda_0 = L\theta$$

which is obviously incorrect since it implies that the lateral distortion of a straight-rolling wheel, which actually must be zero, depends on the choice of the coordinate axes to which θ is referred. Only for the special case where the wheel runs along the reference axis (that is, for $\theta = 0$) is Kantrowitz's theory correct in this respect and in an actual shimmy problem, this is possible only for the case of zero trail; thus Kantrowitz's theory cannot be necessarily expected to give reliable results for trails different from zero. Thus it must be concluded that Kantrowitz's theory is at least of doubtful value for practical shimmy calculations.

To evaluate the Wylie theory, consider the case of steady untilted yawed rolling of a wheel moving parallel to the X-axis. It is obvious that the lateral distortion of the tire λ_0 will depend only on the swivel angle θ ($\theta = \alpha$) and in no manner will depend on the absolute lateral

displacement of the wheel η_0 . On the other hand, the basic equation of Wylie, which in the present terminology is

$$y_0 + LDy_0 = L\theta - l_2 D\theta \quad (6.10)$$

gives for this steady case (where $Dy_0 = D\theta = 0$) the relation $y_0 = L\theta$ or by using equation (2.2)

$$\lambda_0 = L\theta - \eta_0$$

This equation states the obviously incorrect conclusion that the tire distortion is dependent on η_0 or, in other words, that it depends on the choice of the coordinate axes. Thus, only for the special case $\eta_0 = 0$ is Wylie's theory plausible in this respect and this implies that the reference axis must be taken to pass through the path of the wheel. Since this condition is satisfied in an actual shimmy motion only for the special case of zero trail, it must be concluded that Wylie's theory, like Kantrowitz's, can at best be fully valid only for zero trail and that consequently this theory is also of doubtful value for practical shimmy calculations.

Other theories.- In addition to the just discussed theoretical papers dealing particularly with the subject of landing gear shimmy, there exist a number of relevant papers which are either largely of historical interest, which deal

particularly with automobile shimmy problems, which deal only briefly with landing gear shimmy problems, which deal with other tire motion problems such as yawed rolling and verrring off or ground looping, or which deal with the determination of tire stiffness parameters. The reader is referred to the bibliography prepared by Dengler, Goland, and Herrman⁷¹ for a substantially complete listing and brief discussion of most of the papers in this class.

Of particular historical interest among the work not considered here in detail are the work of Broulhiet⁷² and the work of Fromm⁷³. These two investigators independently were apparently the first to recognize the importance of tire lateral distortion and cornering power in regard to the wheel shimmy problem. Taking these factors into account, both authors developed tire motion theories whose kinematic relations correspond to that of approximation C2 of the present paper.

Tabular Classification of Existing Theories

In order to permit easier visualization and comparison of the merits of the various theories discussed, the major

⁷¹ Max Dengler, Martin Goland and Georg Herrman,
op. cit.

⁷² M. G. Broulhiet, op. cit.

⁷³ H. Fromm, op. cit.

assumptions of the various theories of tire motion are collected together in Table I. This table lists the nature of the assumptions made in regard to the primary tire parameters N , K_a , K_λ , ϵ and l_n for each of the theories discussed.

TABLE I

PRIMARY ASSUMPTIONS FOR THE VARIOUS THEORIES OF TIRE MOTION

Theory	N	K _λ	K _α	ε	l ₁	l ₂	l ₃	l _n , n > 3	Remarks
Summary theory	F	F	F	F	F	F	F	F	
Approximation A	F	F	F	F	F	F	F	F	
Approximation B	F	F	F	F	F	F	F	F	
Approximations C1 and C2	F	F	F	F	F	F	F	F	
Approximation D1	F	∞	F	F	F	F	F	F	Assumes laterally rigid tire
Approximation D2	∞	F	∞	F	F	F	F	F	Assumes torsionally rigid tire
Approximation D3	∞	∞	∞	F	F	F	F	F	Assumes laterally and torsionally rigid tire
Schlippe-Dietrich and Rotta	F	F	F	F	F	F	F	F	
Bourcier de Carbon advanced	F	F	F	F	F	F	F	F	
Greidamus	F	F	F	F	F	F	F	F	
Bourcier de Carbon elementary	F	F	F	F	F	F	F	F	
Melzer	F	F	F	F	F	F	F	F	Assumes tire twist angle = swivel angle
Moreland advanced	F	F	F	F	F	F	F	F	Introduces time constant term
Moreland intermediate	F	F	F	F	F	F	F	F	Implies extremely large l ₁ value
Moreland elementary	∞	∞	F	F	F	F	F	F	Assumes laterally and torsionally rigid tire
Temple elementary	F	∞	F	F	F	F	F	F	Assumes laterally rigid tire
Maier	F	∞	F	F	F	F	F	F	Assumes laterally rigid tire
Taylor	∞	F	∞	F	F	F	F	F	Assumes torsionally rigid tire
Kantrowitz	F	F	F	F	F	F	F	F	{ For trail not equal to zero both of these theories can lead to erroneous conclusions
Wylie	F	F	F	F	F	F	F	F	

Note: The symbol F indicates a finite number.

CHAPTER VII

APPLICATION TO WHEEL SHIMMY PROBLEMS

In the preceding chapters of this paper, a set of basic differential equations for the motion of an elastic wheel have been derived and have been compared with the corresponding equations of most of the previously published theories. These comparisons have indicated that, from a mostly qualitative point of view, the summary theory of this paper and the systematic approximations to it incorporate the major merits of the existing theories of tire motion and avoid some of their disadvantages. However, it still remains to investigate how to best apply the theory to specific landing gear problems, to investigate the question of the absolute or quantitative accuracy of the summary theory and of the other theories and, if the summary theory be found satisfactory, to establish the simplest systematic approximation to it which will give reliable information regarding any particular problem in tire motion. The best way to accomplish these various aims appears to be through the discussion of the shimmy of several particular landing gear configurations and such a discussion is given in this chapter. Two particular landing gear configurations are discussed. These two configurations and the reasons for their discussion are described briefly as follows.

Description of Particular Cases Considered

The first landing gear configuration considered, which is designated as Case I, is illustrated in Figure 5. It consists of a rigid landing gear whose only degree of freedom other than tire distortion is rotation of the wheel about an inclined swivel axis. This particular configuration is chosen for the reason that most of the existing experimental data have been obtained for such a configuration. Thus, in connection with this configuration, it is possible to discuss and evaluate the summary theory, its systematic approximations, and the existing theories with respect to agreement with experiment in regard to the various important characteristics of a shimmy motion such as stability boundaries, shimmy frequency and divergence.

The second landing gear configuration considered is the case of an untilted landing gear possessing two degrees of freedom aside from tire distortion. This landing gear configuration, which is illustrated in Figure 6, consists of a wheel free to swivel but not to tilt which turns about a rigid vertical swivel axis, this swivel axis being attached by a spring k to the supporting structure. (This spring is an idealized representation for the lateral flexibility of an actual landing gear strut.) This Case II configuration is discussed for two purposes, first to give an illustration of

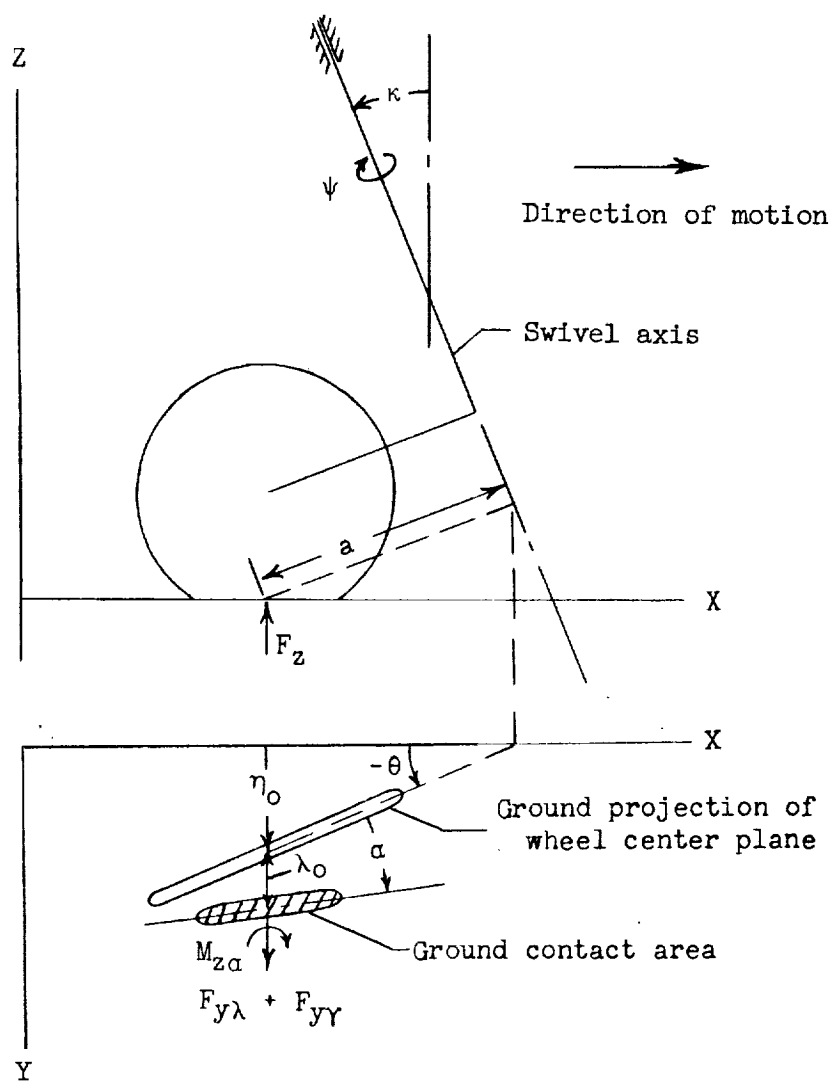


FIGURE 5

CONFIGURATION OF LANDING GEAR FOR CASE I

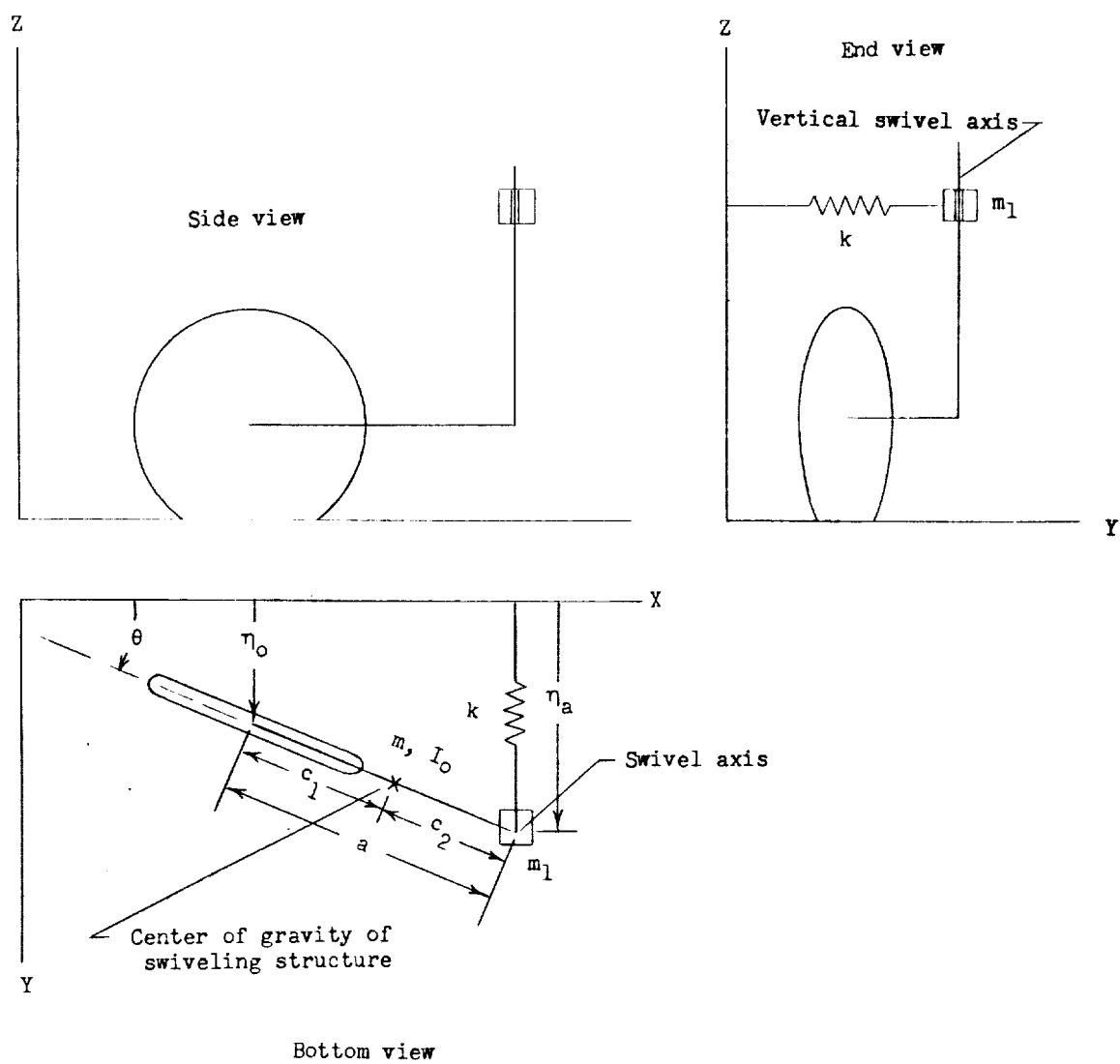


FIGURE 6

CONFIGURATION OF LANDING GEAR FOR CASE II

the effect of structural elasticity on wheel shimmy behavior and second to provide an example which is better suited than Case I for bringing out the relative merits of several of the systematic approximation theories for a case involving structural flexibility.

Case I

General derivation.— In this section, the basic equation of motion is derived according to the summary theory for the special case of an inclined rigid swiveling landing gear (Case I), which is illustrated in Figure 5. This equation of motion could be obtained by making use of the previously derived equations of motion for the completely general case; however, it is simpler to derive it here separately in a slightly different form for this particular problem.

The geometric quantities which enter the discussion of this particular landing gear are indicated in Figure 5. This gear has a swivel axis lying in the XY-plane and is inclined forward from the vertical Z-axis by a constant angle κ (see Figure 5). The perpendicular distance a between the center ground contact point O and the swivel axis is called the trail. The swivel axis is assumed to move with constant velocity v along the X-axis without lateral motion from the XZ-plane.

Rotation of the wheel structure about the inclined swivel axis by an amount ψ results in a component of angular rotation about the vertical axis θ of magnitude

$$\theta = \psi \cos \kappa \quad (7.1)$$

a component of rotation about the X-axis γ (tilt) of magnitude

$$\gamma = -\psi \sin \kappa \quad (7.2)$$

and a lateral deflection η_0 of magnitude

$$\eta_0 = -a \psi \quad (7.3)$$

where all angles except κ are considered small.

The sum of all moments about the swivel axis must equal the inertia reaction $I_{\psi} D_t^2 \psi = I_{\psi} v^2 D^2 \psi$ where I_{ψ} is the moment of inertia of the wheel structure (including the wheel) about the swivel axis. The moments about the swivel axis are assumed to consist of the moments resulting from the previously discussed forces and moments arising from tire distortion and ground loads plus the moments applied to the wheel by the supporting structure which are assumed to consist of a restoring spring of moment $p \psi$ and a linear

damper of moment $gD_t\psi = g\nu D\psi$, where ρ and g are constants. Thus summation of the moments about the swivel axis gives the differential equation

$$\begin{aligned}
 & -[K_\lambda(y_0 - \eta_0) - K_\gamma\gamma]a - F_z \sin \kappa [c_\lambda y_0 + (1 - c_\lambda)\eta_0 - c_\gamma\gamma] + \\
 & K_\alpha \cos \kappa (Dy_0 - \theta) - \tau v^2 \cos \kappa D(y_0 - \eta_0) - \rho\psi - \\
 & g\nu D\psi = I_\psi v^2 D^2\psi
 \end{aligned} \tag{7.4}$$

where the first term is the total ground force due to tire lateral distortion and tilt (see equations (3.1) and (3.8)) times its moment arm a ; the second term is the vertical force times its moment producing fraction $\sin \kappa$ times its moment arm (see equation (3.9)); the third term is the moment about the Z-axis due to tire twist (see equation (3.5)) corrected by $\cos \psi$ for the component about the swivel axis; the remaining terms on the left hand side represent the gyroscopic torque due to lateral tire distortion (see equation (3.12)); the spring restoring moment and the linear damper moment. Now by making use of equations (7.1) to (7.3), equation (7.4) can be written in the form

$$A_1 D^2\psi + A_2 D\psi + A_3\psi + B_1 Dy_0 + B_2 y_0 = 0 \tag{7.5a}$$

where

$$\begin{aligned}
 A_1 &= I_\psi v^2 \\
 A_2 &= a\tau v^2 \cos \kappa + g v \\
 A_3 &= a^2 K_\lambda + K_\alpha \cos^2 \kappa + \rho + \rho_\kappa \\
 B_1 &= -K_\alpha \cos \kappa + \tau v^2 \cos \kappa \\
 B_2 &= aK_\lambda + c_\lambda F_z \sin \kappa
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \end{aligned}} \right\} (7.5b)$$

and

$$\rho_\kappa = aK_\gamma \sin \kappa - aF_z \sin \kappa + ac_\lambda F_z \sin \kappa + c_\gamma F_z \sin^2 \kappa \quad (7.5c)$$

The general relation between ψ and y_0 for this case is found by substituting for η_0 , γ and θ , according to equations (7.1) to (7.3) in the general kinematic equation (2.20). Thus

$$y_0 + \sum_{n=1}^{\infty} l_n D^n y_0 = -a\psi + l_1 \psi \cos \kappa + \frac{\xi L h}{r} \psi \sin \kappa$$

or abbreviating

$$\sigma = 1 + \frac{\xi L h}{r l_1} \tan \kappa \quad (7.6)$$

and rearranging

$$(\sigma l_1 \cos \kappa - a) \Psi = y_0 + \sum_{n=1}^{\infty} l_n D^n y_0 = \sum_{n=0}^{\infty} l_n D^n y_0 \quad (7.7)$$

since $l_0 = 1$. Differentiating this result gives

$$(\sigma l_1 \cos \kappa - a) D \Psi = \sum_{n=0}^{\infty} l_n D^{n+1} y_0 = \sum_{n=1}^{\infty} l_{n-1} D^n y_0 \quad (7.8)$$

and similarly

$$(\sigma l_1 \cos \kappa - a) D^2 \Psi = \sum_{n=2}^{\infty} l_{n-2} D^n y_0 \quad (7.9)$$

Substitution of these relations into equation (7.5) and multiplication through by $\sigma l_1 \cos \kappa - a$ gives

$$A_1 \sum_{n=2}^{\infty} l_{n-2} D^n y_0 + A_2 \sum_{n=1}^{\infty} l_{n-1} D^n y_0 + A_3 \sum_{n=0}^{\infty} l_n D^n y_0 + \\ B_1 (\sigma l_1 \cos \kappa - a) D y_0 + B_2 (\sigma l_1 \cos \kappa - a) y_0 = 0$$

Finally after adding all terms of like order, substituting $N = l_1 K_\lambda$ (equation (4.9)), substituting for some of the A 's and using equation (7.6), there results the equation

$$\sum_{n=0}^{\infty} F_n D^n y_0 = 0$$

where

$$F_0 = \sigma a N \cos \kappa + K_a \cos^2 \kappa + \rho + \rho_\kappa + u_\kappa$$

$$F_1 = a^2 N + a' K_a \cos \kappa + \rho l_1 + \rho_\kappa l_1 + g v + \sigma l_1 v^2 \cos \kappa$$

$$F_2 = A_1 + A_2 l_1 + A_3 l_2$$

$$F_n = A_1 l_{n-2} + A_2 l_{n-1} + A_3 l_n; \quad n > 2$$

and

$$u_\kappa = c_\lambda F_z \sin \kappa (\sigma l_1 \cos \kappa - a)$$

$$a' = a + (1 - \sigma) l_1 \cos \kappa = a - \frac{t L h}{r} \sin \kappa$$

(7.10)

Equation (7.10) provides the general differential equation of free motion for the system of Case I according to the summary theory. The corresponding equations for the systematic approximations A to D3 can be easily obtained from this equation by setting the appropriate l_n 's and t , K_λ , K_a , and N equal to zero or infinity according to the procedures outlines in Chapter V. For example, for approximation C2 the differential equation is obtained by letting $l_n = 0$ for $n > 1$ and $t = 0$, in equation (7.10). The following differential equation is thus obtained for approximation C2.

$$E_0 D^3 y_0 + E_1 D^2 y_0 + E_2 D y_0 + E_3 y_0 = 0$$

where

$$E_0 = I_v v^2 l_1$$

$$E_1 = I_v v^2 + (a r v^2 \cos \kappa + g v) l_1$$

$$E_2 = a^2 N + a K_a \cos \kappa + \rho l_1 + \rho_\kappa l_1 + g v + l_1 r v^2 \cos^2 \kappa$$

$$E_3 = a N \cos \kappa + K_a \cos^2 \kappa + \rho + \rho_\kappa + u_{\kappa 1}$$

and

$$u_{\kappa 1} = c_\lambda F_z (l_1 \cos \kappa - a) \sin \kappa$$

(7.11)

Stability of motion.— Now having established the basic equations of motion for the case of a rigid swiveling landing gear, attention is directed next to the meaning of these equations with respect to their predictions of the shimmy behavior of the landing gear. However, before going into this subject in detail, it may be useful to discuss briefly what sort of information is desired about the motion of a landing gear. Basically, the most important question is to determine whether or not the motion is stable, that is, does the wheel tend to move in a straight line (with decaying

shimmy oscillations or decaying aperiodical motion) or does the tire tend to move laterally out from its rectilinear course (with divergent shimmy oscillations or divergent aperiodical motion). To answer this question of stability for linear systems, the analytic methods of Routh⁷⁴ or Hurwitz⁷⁵ or graphical methods similar to those introduced by Nyquist^{76,77} are available. Any of these methods will provide, for most cases, a procedure for determining whether any particular combination of landing gear parameters and rolling velocity is stable or unstable.

In general, for complicated problems, rather than investigate the stability of a landing gear by these methods for all possible conditions, it may be more convenient and sometimes more valuable to draw various types of stability diagrams describing the system in question. For example, for Case I, a typical experimental type of stability diagram is shown in Figure 7 which presents boundaries between the

⁷⁴ Edward J. Routh, Dynamics of a System of Rigid Bodies, Part II. Sixth edition; New York: The MacMillan Co., 1905, 484 pp.

⁷⁵ E. A. Guillemin, The Mathematics of Circuit Analysis. New York: John Wiley and Sons, Inc., 1949, 590 pp.

⁷⁶ H. Nyquist, "Regeneration Theory," Bell System Technical Journal, Vol. 11, Jan. 1932, Pp. 126-141.

⁷⁷ W. Frey, "A Generalization of the Nyquist and Leonhard Stability Criteria," Brown Boveri Review, Vol. 33, No. 3, March 1946, Pp. 59-65.

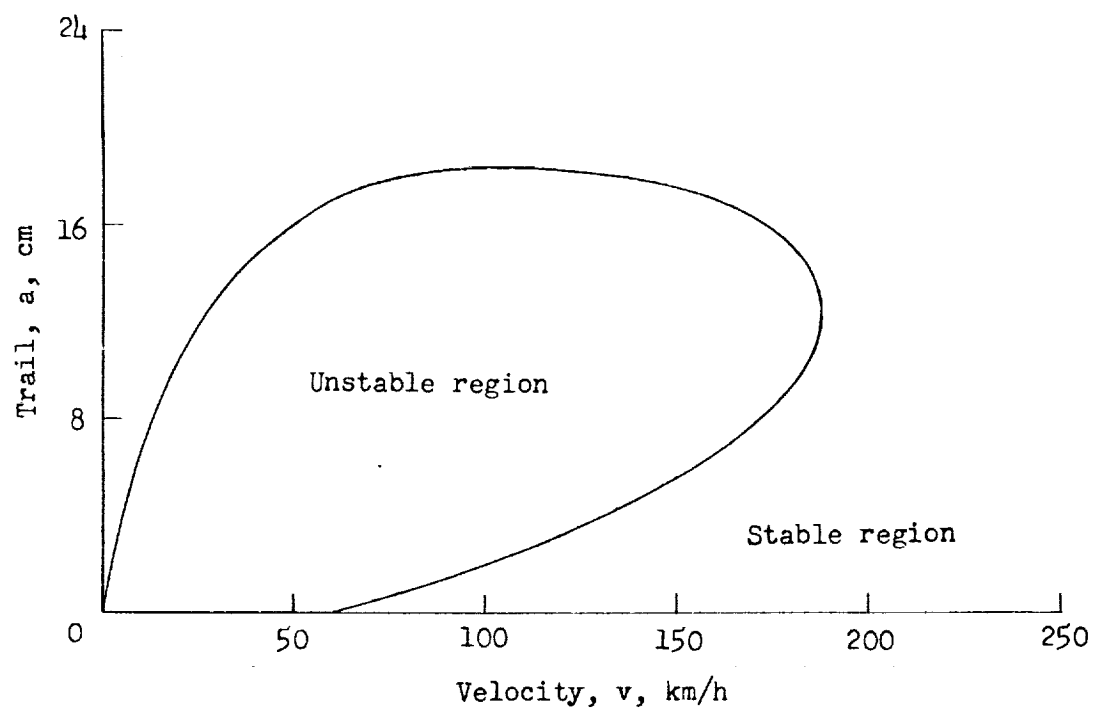


FIGURE 7

EXPERIMENTAL STABILITY BOUNDARY FOR A 29-CENTIMETER
DIAMETER TIRE TESTED BY SCHRODE

regions of stability and instability as functions of trail and rolling velocity for a specific landing gear model. Another useful type of stability diagram for some problems might be a plot of boundaries between stable and unstable regions as functions of damping moment and rolling velocity.

To determine these stability boundaries, use is made of the well known fact that the motion of a linear system can change from a stable to an unstable condition only where the motion is purely oscillatory, in terms of ψ , of the form

$$\psi = \psi_m e^{i\omega_1 x} \quad (7.12)$$

or where the motion is purely uniform, of the form

$$\psi = \psi_m \quad (7.13)$$

Thus all possible stability boundaries can be obtained by directly substituting expressions of the form of equations (7.12) and (7.13) into the basic differential equations. In connection with the question of what form of the differential equation to use, it is of some importance to note that the final form where the equation is expressed in terms of one variable is often not the most convenient form to use. For

example, for Case I, the purely oscillatory boundaries are most advantageously obtained by using the equations (7.5) and (7.7) with the substitutions

$$\left. \begin{aligned} \psi &= \psi_m e^{i v_1 x} \\ y_0 &= y_{0m} e^{i(v_1 x + a_1)} \end{aligned} \right\} (7.14)$$

The advantage in this particular choice arises from the fact that it leads to two algebraic equations, one of which does not include the damping parameter g . This isolation of the parameter g usually slightly eases the mathematical labor of solving for the purely oscillatory boundaries.

The equations governing the stability boundaries for Case I for the summary theory and for the systematic approximations are listed in the Appendix.

Comparison and evaluation of the summary theory and its systematic approximations.— The dual object of the present section is (1) to further assess the value of the summary theory by comparisons between the predictions of this theory and the available experimental data for Case I conditions and (2) to determine by comparison of the relative predictions of the summary theory and its systematic approximations, what is the simplest satisfactory systematic approximation to the summary theory. Discussion of the previously published

theories, as applied to Case I conditions, is contained in a later section.

For convenience, the following discussion is divided into separate considerations of stability boundary conditions and unstable shimmy conditions.

Stability boundary conditions: The present subsection deals with a discussion of theoretical and experimental stability boundary conditions inasmuch as they are influenced by the tire parameters l_n ($n = 1, 2, \dots$), ξ , N , and τ . In the major part of this discussion, the type of stability boundaries considered are the type obtained by plotting curves of trail against rolling velocity for those trail conditions separating regions of stability and instability. The general shapes of these stability boundaries for Case I, according to the summary theory and the systematic approximation theories A to D3, are sketched in Figure 8 for the special condition of no damping or gyroscopic moments ($g = \tau = 0$). It is seen that the summary theory and approximations A to C2 each predict that at high speeds the motion is stable for large trails and unstable for small trails; the horizontal boundary line is the same for each case, and is generally located at a trail roughly equal to the tire radius. (This boundary is theoretically completely independent of the spring restoring moment ρD_t and is relatively independent of swivel axis inclination κ .)

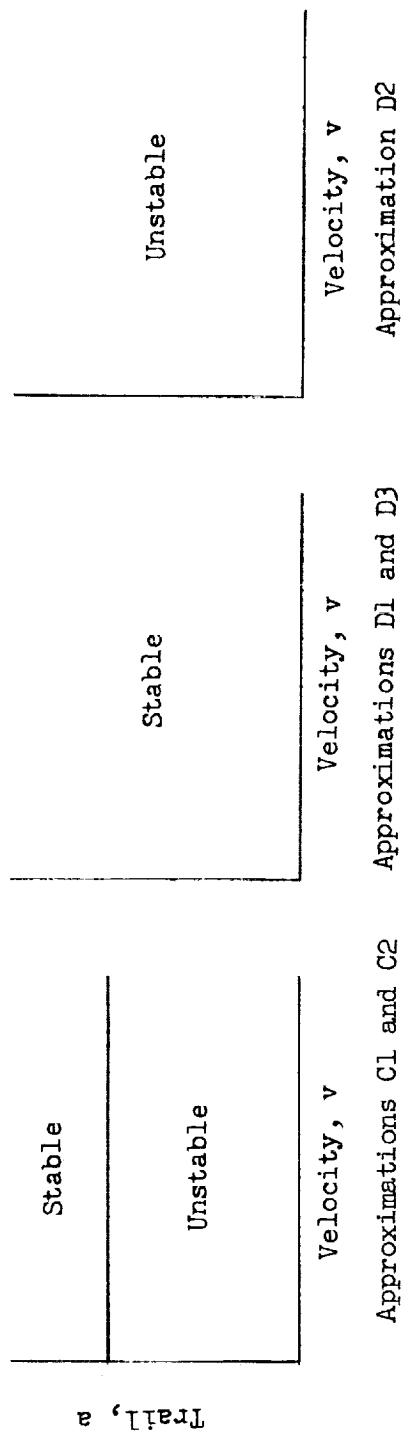
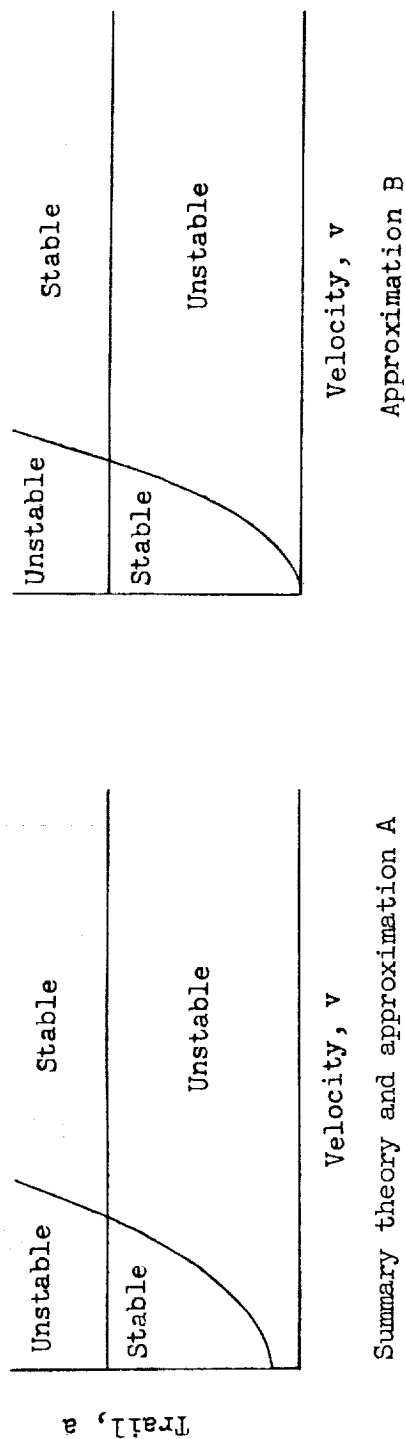


FIGURE 8

QUALITATIVE COMPARISON OF THE STABILITY BOUNDARY PREDICTIONS FOR
CASE I ACCORDING TO THE SUMMARY AND SYSTEMATIC APPROXIMATION
THEORIES NEGLECTING DAMPER AND GYROSCOPIC MOMENTS

Approximations D1, D2, and D3 fail to predict this boundary. Also these three approximations, together with approximations C1 and C2, fail to predict any effect of rolling velocity on the low speed stability boundaries while the higher theories demonstrate that for sufficiently small speeds, the motion becomes stable for all small trails according to approximation B and for most of the small trail regions according to the higher theories. Also at low speeds and large (usually impractical) trails, the higher theories (B and above) indicate that the motion becomes unstable at sufficiently small speeds. The effects of the omitted damper and gyroscopic moment terms would be to reduce the size of the regions of instability.

(a) Effect of higher l_n terms: As a first test of the summary theory and its systematic approximations, there are available the experimental data of Schlippe and Dietrich,^{78,79,80} which were obtained with a small model landing gear equipped with a 26 cm (10 in.) diameter pneumatic tire. This model landing gear was tested at relatively low speed conditions where the higher l_n terms (l_2, l_3, \dots)

⁷⁸ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

⁷⁹ B. von Schlippe and R. Dietrich, "Das Flattern eines bepneuten Rades," op. cit.

⁸⁰ B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

are of some importance; consequently, these data provide an opportunity for testing the relative and absolute validity of the summary theory and the higher approximation A to C2 (which differ essentially only by their inclusion or omission of the higher l_n terms).

The basic landing gear and tire constants for the Schlippe-Dietrich model, which was tested only in the unutilized condition ($\kappa = 0$), as taken from Schlippe and Dietrich's papers, are as follows

$$\kappa = \rho = g = 0$$

$$I \approx 0.53 + 0.0025a^2 \text{ cm-kg-sec}^2$$

$$L = 10 \text{ cm}$$

$$N = 640 \text{ kg/rad}$$

$$K_a = 3040 \text{ cm-kg/rad}$$

$$K_\lambda = 45 \text{ kg/cm}$$

The quantities l_1 , h and the higher l_n 's were calculated from the previously discussed relations $l_1 = N/K_\lambda$, $h = l_1 - L$, and $l_n = (nL + h)^{n-1}/n!$ (see equations (4.9) and (2.20)).

The experimental data obtained by Schlippe and Dietrich for the model are shown in Figures 9 and 10 together with the corresponding predictions of the summary theory and the systematic approximations A to C2. (Also shown are the predictions of the theory of Schlippe and Dietrich which are discussed in a later section.) Figure 9 presents stability boundary plots of trail against velocity and Figure 10 presents the frequency at these stability boundaries as a function of velocity. No theoretical curves are shown on these Figures for approximations D1, D2 and D3 since these approximations are too crude to give any detailed information for this problem; they either predict completely stable or completely unstable motion for all positive trails (see Figure 8). The equations used to calculate the theoretical curves in these two Figures are given in the appendix. In these calculations, the gyroscopic torque term involving τ has been neglected since τ is unknown for these data. While a rough value of τ could perhaps be estimated, such a dubious estimate did not appear necessary since the term involving τ , according to any reasonable estimate of τ , would be of no importance in the velocity range of these experimental data.

To first compare the theoretical curves in Figures 9 and 10, it is observed that approximation A gives a boundary very close to that of the summary theory. Approximation B

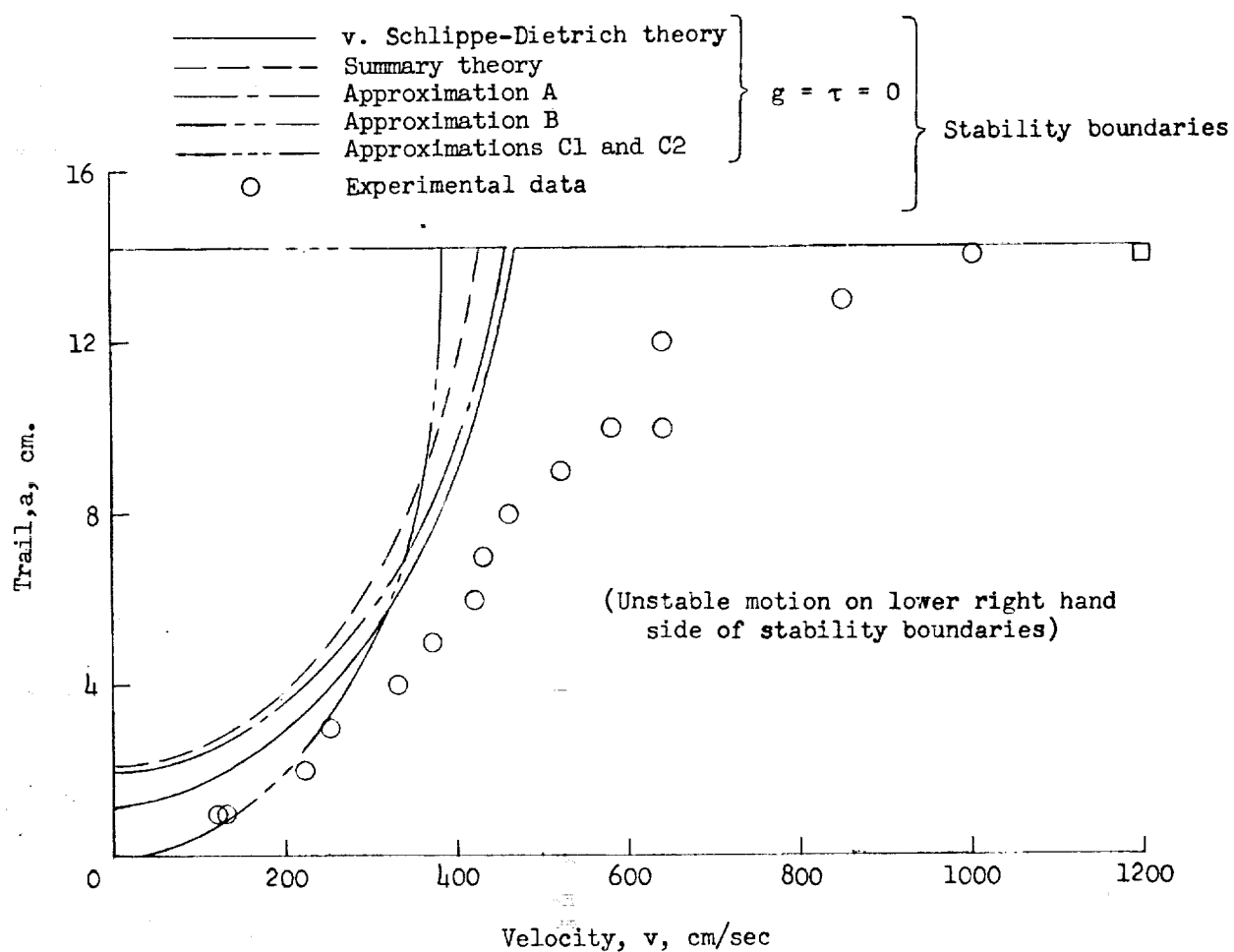


FIGURE 9

COMPARISON OF THEORETICAL AND EXPERIMENTAL PREDICTIONS OF THE STABILITY BOUNDARIES FOR THE TEST SYSTEM OF SCHLIPPE AND DIETRICH

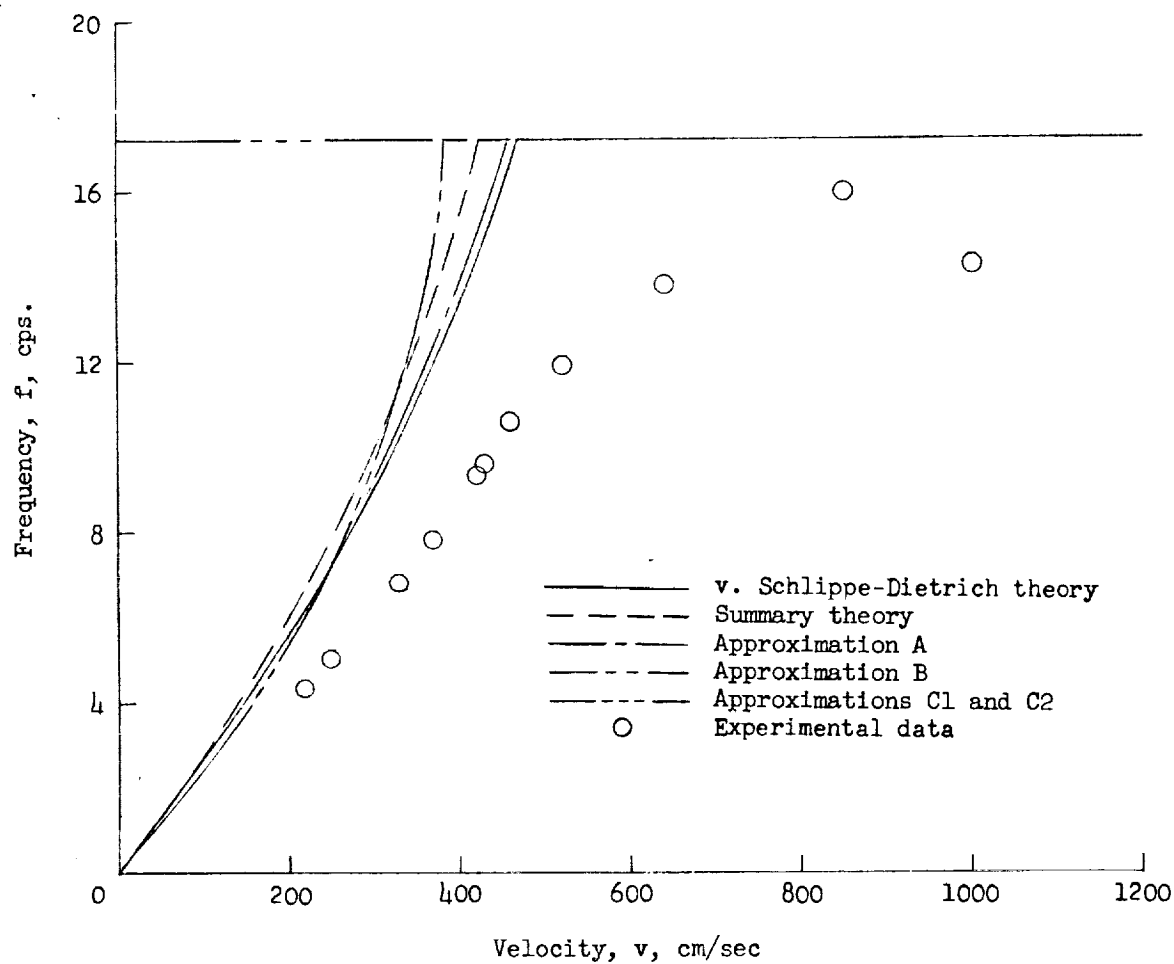


FIGURE 10

COMPARISON OF THEORETICAL AND EXPERIMENTAL SHIMMY FREQUENCIES ON THE STABILITY BOUNDARY FOR THE TEST SYSTEM OF SCHLIPPE AND DIETRICH

does not give as close agreement but it is still fairly good and, more importantly, for most of the trail range, the difference between approximation B and the summary theory is small beside the difference between the summary theory and the experimental data. As was previously noted, approximations C1 and C2 (which are identical for the present condition of $\kappa = 0$) predicts a trail-velocity stability boundary which is independent of velocity so that this approximation is an inadequate representation of the summary theory at low velocities. However, at high speeds, approximations C1 and C2 give the same stability boundary and frequency as the higher approximations.

As a further aid in comparing the different systematic approximations with the summary theory, Figure 11 presents a plot of the linear damping coefficient γ required to stabilize the motion of the Schlippe-Dietrich model at a medium trail of 7 cm as calculated according to the summary theory and the various systematic approximations (the equations used are presented in the Appendix). This Figure confirms the conclusions drawn from the previous Figures 9 and 10, namely, that approximation A is a very good representation of the summary theory and that approximation B is also a good representation of the summary theory. However, more importantly, this Figure demonstrates that approximations C1 and C2 also give a fairly good representation of the summary

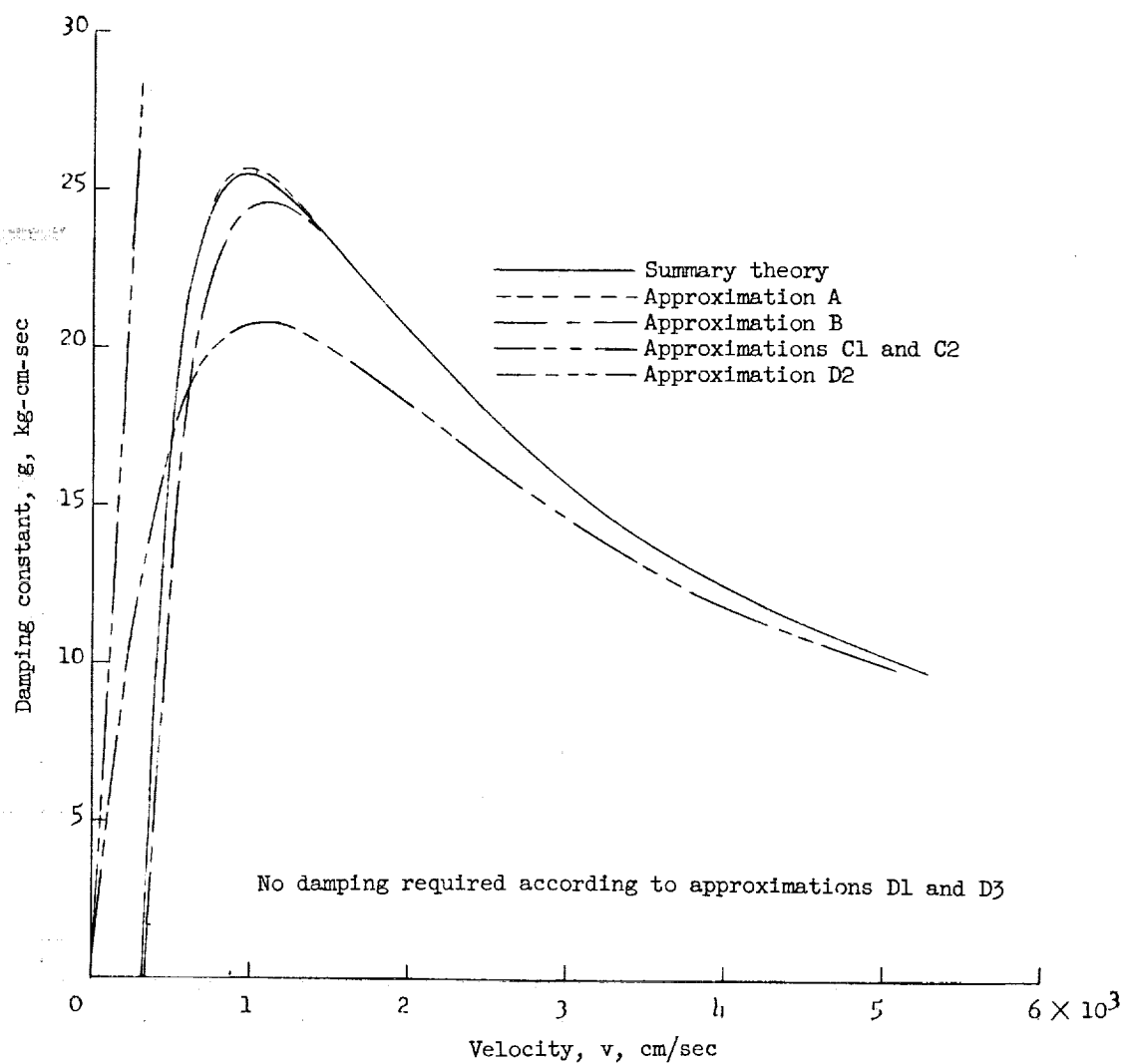


FIGURE 11

THEORETICAL CALCULATIONS OF THE DAMPING REQUIRED TO
 STABILIZE THE MOTION OF THE TEST SYSTEM OF SCHLIPPE AND DIETRICH
 AT A TRAIL OF 7 CENTIMETERS

theory with respect to prediction of the maximum amount of damping (that is, the maximum value of g) required for stabilizing the motion. Approximations D1, D2, and D3 are seen to give inadequate representations of the summary theory.

The preceding conclusions are, of course, only proven to be valid for the specific conditions of the Schlippe-Dietrich model tests. However, it is believed that these conclusions are probably valid for most practical rolling conditions.

To next consider the correlation between theory and experiment for the Schlippe-Dietrich test conditions, it is noted that the experimental stability boundary in Figure 9 is of the same general shape as that given by the summary theory and approximations A and B, but that it lies to the right of the theoretical curves thus indicating that the experimental system is more stable than the theoretical system. Similarly, the experimental frequency-velocity curve in Figure 10 falls below the theoretical curves. These discrepancies are perhaps a result of the neglect of hysteresis damping in the calculation of the theoretical curves.

(b) Effect of l_1 : The next test of the summary theory will be made by making use of the experimental data of Melzer,⁸¹ who performed a series of model tests with an

⁸¹ M. Melzer, op. cit.

Thus the results of the preceding comparison do indicate that there exists a range of rolling speeds in which the kinematic equation of the summary theory, as well as of approximations A to C2 is reasonably correct (except possibly for the as yet not evaluated and not too important terms involving ϵ).

In regard to the question as to whether these calculations hold for the entire practical range of rolling speeds, it can be said with safety that the range of velocity for which the theory gives good agreement with Melzer's model data corresponds to full scale conditions somewhere inside the practical rolling speed range and possibly covering much of the practical range. However, the preceding comparison definitely does not prove anything about the adequacy of the summary theory for small velocities or for the highest velocities which may be encountered in practice.

Further confirmation of the preceding conclusions are provided by the experimental data of Schrode,⁸² who performed tests similar to the just discussed tests of Melzer, for realistic pneumatic tires as large as 39 cm (15 in.) in diameter, as compared to the small 7 cm (3 in.) in diameter solid rubber tire tested by Melzer, and obtained trail-velocity stability boundary plots of the type illustrated in

⁸²H. Schrode, op. cit.

Figure 7. These stability boundary plots indicate the same result as Melzer's data, namely, that there exists a range of velocity in which the motion is stable above a certain critical trail a_c and unstable below it. While it is not possible to quantitatively check the theoretical stability boundary equation $a_c = l_1$ for Schrode's data since Schrode provides no information suitable for accurately evaluating l_1 , some qualitative confirmation may be found since the quantity l_1 always appears to be of the order of magnitude of the tire radius r and, for Schrode's data, a_c is found to be of this same order of magnitude (for example, see Figure 7). Thus Schrode's experimental data appear to confirm the previously drawn conclusion that there exists a velocity range in which the kinematic equations of the summary theory and approximations A to C2 are valid.

Dietz and Harling⁸³ have presented some similar stability boundary curves which also confirm the foregoing conclusions.

(c) Effect of ξ : Some insight into the effect of the tilt parameter ξ can be obtained by an examination of the effects of swivel axis inclination κ on the stability boundaries according to the predictions of approximation C1 for the condition where damping, spring restoring and

⁸³

O. Dietz and R. Harling, op. cit.

gyroscopic moments are neglected ($g = p = r = 0$) in order to isolate the effects of inclination. (These assumptions appear to be justified for the experimental conditions to be discussed in this section.) Under these assumptions, one theoretical stability boundary is given by the equation

$$a_c = l_1 \cos \kappa + \frac{eLh}{r} \sin \kappa \quad (7.15)$$

Experimental data suitable for testing this relation have been obtained by Dietz and Harling⁸⁴ for an inclination range $-20^\circ < \kappa < 20^\circ$ for one constant velocity condition. These experimental data, some of which has to be slightly extrapolated from Dietz and Harling's data, are shown in Figure 12 together with the predictions of equation (7.15) for values of ξ equal to 0 and 1. While Dietz and Harling did not supply the values of L , h and l_1 needed for calculations, the assumed values indicated on the Figure are probably accurate enough to justify the following more or less qualitative conclusions. (The value of l_1 was chosen such as to make the calculated and experimental values agree for the case $\kappa = 0$.) It is noted that the experimental variations and the theoretical variations for $\xi = 0$ are in fairly good agreement and also that these two variations

⁸⁴ Ibid.

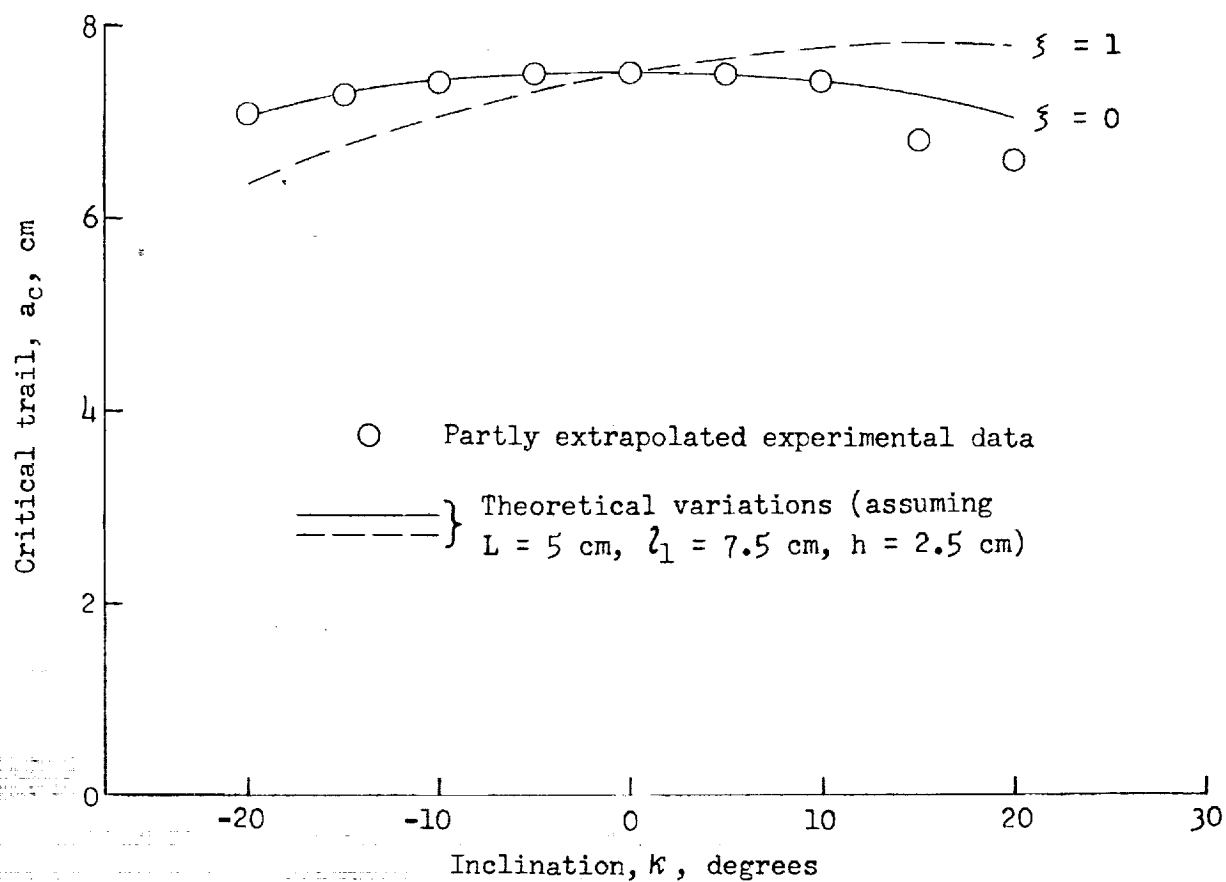


FIGURE 12

INFLUENCE OF SWIVEL AXIS INCLINATION ON THE STABILITY BOUNDARY FOR A
 12-CENTIMETER DIAMETER TIRE. $F_z = 6.25 \text{ kg}$, $v = 19 \text{ km/hr}$

are more or less symmetrical with respect to positive and negative values of κ . On the other hand, the theoretical curves for $\xi > 0$ such as the indicated curve for $\xi = 1$ will all be unsymmetrical. Thus, it appears that ξ is probably close to zero. In this connection, it might be noted that Greidanus' theory, which is the only known theory using a ξ -type term, implies a value $\xi > 1$ (compare equations (6.3) and (6.4)).

(d) Effect of cornering power N : As a rough check on the variation of the tire cornering power N under dynamic conditions, there are available experimental frequency data obtained by Melzer⁸⁵ in connection with his previously mentioned tests with an uninclined ($\kappa = 0$) model landing gear equipped with a 7 cm (3 in.) in diameter solid rubber tire. For the higher velocity conditions of Melzer's tests, the predictions of the summary theory and approximations A to D1 lead to the frequency equation

$$f = \frac{1}{2\pi} \sqrt{\frac{aN + K_a + \rho}{I \downarrow}} \quad (7.16)$$

for an uninclined and undamped landing gear, that is, for $\kappa = \tau = g = 0$. (Inclusion of the effect of finite τ into

⁸⁵ M. Melzer, op. cit.

this equation would not significantly alter this equation for the test conditions to be discussed here.) Some of Melzer's experimental data are compared with the predictions of this equation in Table IV for the condition $\rho = 0$. The experimental data shown represent Melzer's data for the highest velocity condition tested. The theoretical and experimental values shown are seen to be in fair agreement. However, the experimental values do seem to be definitely somewhat smaller than the corresponding theoretical values. This discrepancy is believed to be largely due to the fact that these experimental tests were not conducted at sufficiently small values of shimmy amplitude for the assumptions of a linearized theory to be valid. Specifically, all of Melzer's frequency data were obtained for maximum swivel angles of 5° or larger. (The data shown in Table IV correspond to the condition of a 5° maximum swivel angle.) Moreover, Melzer's data indicate that there is a fairly definite decrease in shimmy frequency with increasing maximum swivel angle. A sample plot of Melzer's data illustrating this effect is given in Figure 13. Also shown is the theoretical calculation which is valid only for zero maximum swivel angle. It is seen that, if allowance is made for a certain amount of experimental error, extrapolation of the experimental data to $\theta_m = 0$ could be considered to lead to confirmation of the theory. It should be noted, however,

TABLE IV
SHIMMY FREQUENCY TEST DATA OBTAINED BY MELZER
FOR THE CONDITION $\rho = 0$

F_z , kg	2.8		3.6		
a/l_1	0.47	0.78	0.44	0.73	0.88
$f_{\text{calculated}}$, cps	3.8	4.5	4.0	4.8	5.1
$f_{\text{experimental}}$, cps	3.3	3.5	2.7	4.1	4.7

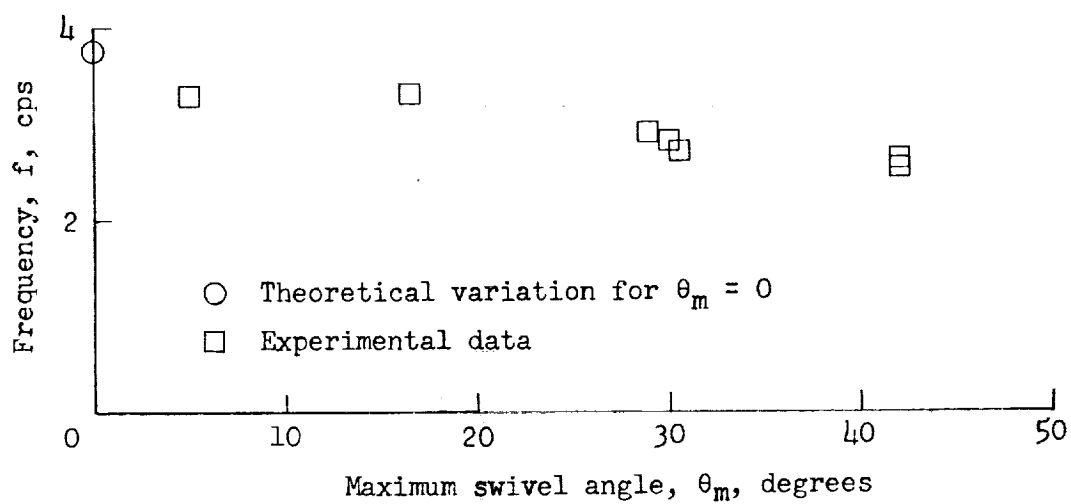


FIGURE 13

INFLUENCE OF SHIMMY AMPLITUDE ON THE SHIMMY FREQUENCY

that much of the rest of Melzer's data, while not necessarily disputing this conclusion, do not so clearly support it. Also it should be noted that plots of the type of Figure 13 are of limited significance since each test point shown corresponds to a different rolling velocity. In view of these considerations, the only reasonable conclusion that can be reached appears to be that Melzer's data roughly confirm the theoretical frequency and do not conclusively dispute its quantitative accuracy.

Melzer also conducted frequency tests on the same model with an additional strong restoring spring (spring stiffness several times the tire torsional stiffness). A comparison of theoretical and experimental frequencies for this test is shown in Table V. The much better agreement obtained for this case is explained by the predominant influence of the spring restoring moment since for large ρ the model system approaches the condition of a simple torsional oscillator of moment of inertia I_{\downarrow} and spring constant ρ for which condition the well known frequency equation is

$$2\pi f = \sqrt{\rho/I_{\downarrow}}.$$

In order to assess the significance of the preceding comparisons, first consider the quantities involved in the theoretical equation (7.16), namely, a , N , K_a , ρ , and I_{\downarrow} . The quantities a , ρ , and I_{\downarrow} are easily measured constants

TABLE V
SHIMMY FREQUENCY TEST DATA OBTAINED BY MELZER
FOR THE CONDITION $\rho \neq 0$

F_z , kg	2.0	2.8		3.6	
a/l_1	0.77	0.69	0.86	0.69,	0.86
$f_{\text{calculated}}$, cps	5.2	5.4	5.7	5.5	5.8
$f_{\text{experimental}}$, cps	4.9	5.45	5.9	5.8	5.85

and for most of Melzer's data, K_a is much smaller than aN ; therefore, the preceding fair agreement between theory and experiment indicates that the quantity N , the tire cornering power, which was considered to be a constant in the preceding calculations, actually does not vary extremely with rolling velocity and shimmy frequency, at least not for Melzer's test conditions.

(e) Effect of gyroscopic torque: The next question to be considered is the influence of the gyroscopic torque resulting from tire lateral distortion. All pertinent experimental data obtained at very high speeds (for example, see Figure 7) demonstrate that at sufficiently high speeds, the previously discussed conclusion that high-speed motion is unstable for trails less than l_1 is no longer valid. Instead at these very high speeds, the experimental data show that instability at any given positive trail ceases above a certain critical velocity. The existence of this critical velocity will now be shown to result, at least in part, from the gyroscopic action which was previously included only in Kantrowitz's theory,⁸⁶ but was not mentioned there specifically. The simplest systematic approximation which adequately provides for this effect is approximation C2. In order to isolate the gyroscopic effect, consider the

⁸⁶

Arthur Kantrowitz, op. cit.

special condition of no tilt ($\kappa = 0$) and no spring restoring force ($\rho = 0$) or damper ($g = 0$). For this condition the equation for the stability boundary of approximation C2 (or C1) reads

$$(I_{\downarrow} v_c^2 + \underline{a \tau v_c^2 l_1})(a^2 N + a K_a + \underline{l_1 \tau v_c^2}) = I_{\downarrow} v_c^2 l_1 (a N + K_a) \quad (7.17)$$

where the underlined terms are the gyroscopic terms. For the computation of the critical velocity v_c this equation may be simplified still further if it is realized that the quantity $a \tau l_1$ is small beside the moment of inertia I_{\downarrow} about the swivel axis; hence, for an approximate calculation, the term $a \tau v_c^2 l_1$ can be omitted. Then solution of equation (7.17) for the critical velocity v_c above which the system is stable yields the expression

$$v_c = \sqrt{\frac{(l_1 - a)(a N + K_a)}{l_1 \tau}} \quad (7.18)$$

(which is observed to give an infinite critical velocity for zero gyroscopic action ($\tau = 0$)).

The only available experimental data containing enough information on the necessary tire constants for checking the validity of equation (7.18) is Melzer's data⁸⁷

⁸⁷ M. Melzer, op. cit.

and even this data does not provide the required gyroscopic moment; therefore, it can only be crudely estimated as follows. The mass of the tire will be of the order of magnitude $w_1 2\pi(r - r_4) \pi r_4^2$ where r is the tire overall radius, r_4 the tire torus radius and w_1 the average tire density. The moment of inertia will be the mass times the radius of gyration r_g squared; thus τ (see equation (3.13)), with $\tau_1 = 1/2$ according to Kantrowitz,⁸⁸ becomes

$$\tau = \frac{\pi^2 w_1 r_4^2 (r - r_4) r_g^2}{r(r + r_3)}$$

For the usual tire $r_4 \approx 0.3r$, r_3 is slightly smaller than r , say $r_3 \approx 0.9r$, and r_g is probably around $0.8r$. Then to a crude approximation $\tau \approx 0.21 w_1 r^3$. For Melzer's solid rubber tire $r = 3.5$ cm and w_1 is probably around 10^{-6} kg-sec²/cm⁴ (specific gravity of one), thus $\tau \approx 10^{-5}$ kg-sec²/cm. Critical velocities calculated with this value of τ from equation (7.18) are compared in Figure 14 with some of Melzer's experimental data for one test condition at various values of a/l_1 . The calculated and experimental values of critical velocity are seen to be of the same order of magnitude. Since neglect of the gyroscopic moment gives theoretically an infinite critical

⁸⁸ Arthur Kantrowitz, op. cit.

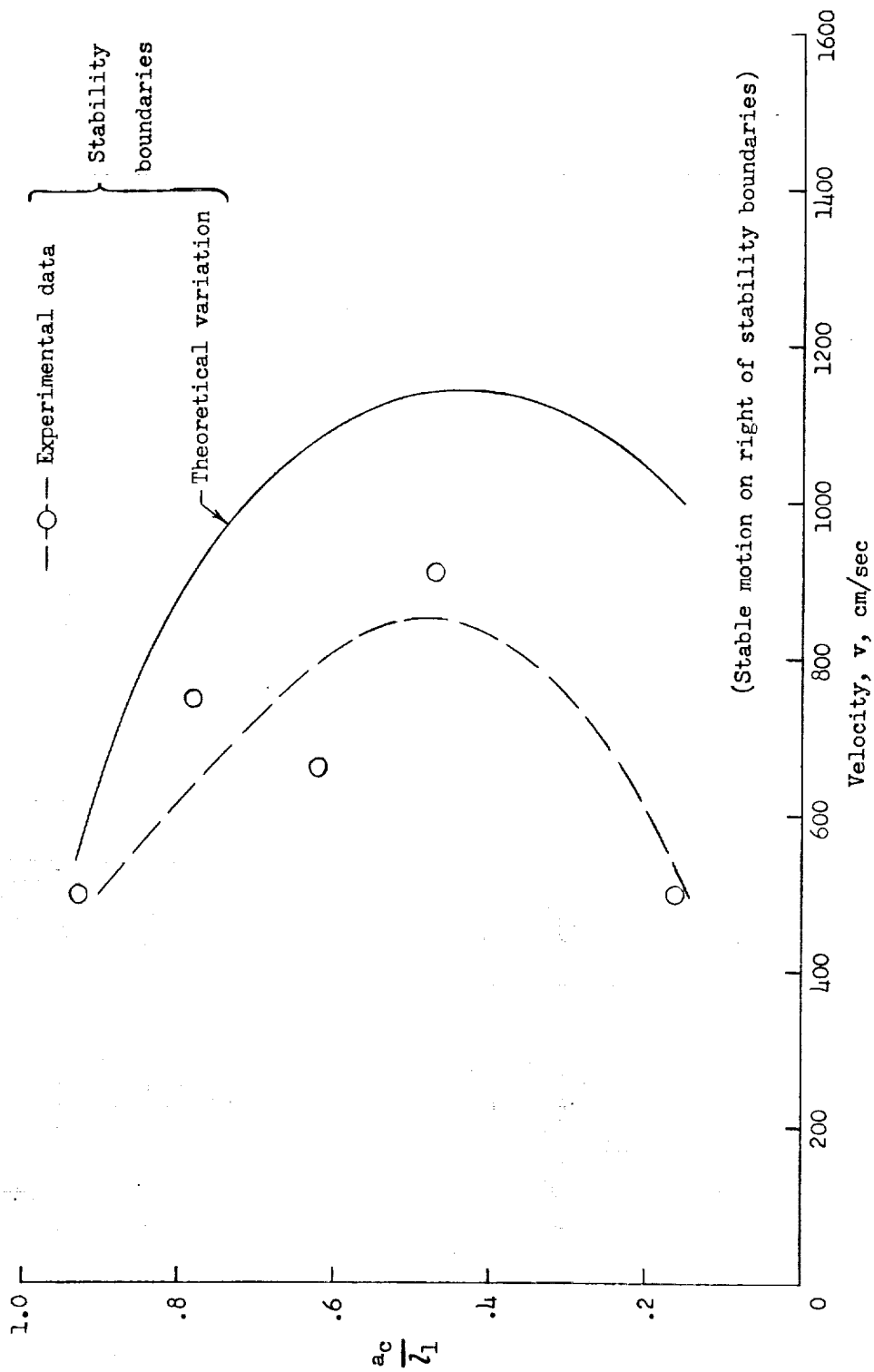


FIGURE 14

COMPARISON OF THEORETICAL AND EXPERIMENTAL
 VARIATIONS OF CRITICAL TRAIL WITH ROLLING VELOCITY.
 $r = 3.5$ cm, $l_1 = 3.22$ cm, $N = 12.3$ kg, $K_a = 5.9$ kg-cm

velocity, this agreement indicates that the gyroscopic moment is an important factor in producing stability at high velocities. It is also of interest to note that the theoretical calculation is conservative, that is, the unstable region is overestimated. In regard to quantitative agreement between theory and experiment, the agreement is fair but far from excellent. One probable reason for some of the indicated disagreement is the relatively crude procedure used for estimating the parameter τ .

In concluding this discussion of gyroscopic torque, it should be noted that for the case of a rigid landing gear, the critical design condition (velocity at which shimmy is most intense) occurs at low rolling speeds where the gyroscopic moment is insignificant. Thus, the inclusion of this gyroscopic moment in the theory is somewhat of purely theoretical interest (at least for Case I) and probably could be safely omitted in practical design calculations.

Unstable shimmy conditions: As a further overall check of the summary theory and its systematic approximations there are available the experimental data of Kantrowitz⁸⁹ for unsteady shimmy conditions.

In the case of unsteady shimmy motion, the significant features of the motion are the frequency and divergence

of the oscillation, where the divergence and frequency are simply the real and imaginary parts of the roots of the characteristic algebraic equation corresponding to the differential equation in question. Kantrowitz has made measurements of these quantities for a 4-inch diameter model tire at inclination angles of $\alpha = 5^\circ$ and 20° with corresponding trails of about $0.08r$ and $0.31r$, respectively. His experimental results for $\alpha = 5^\circ$ are presented in Figure 15 together with corresponding theoretical calculations made according to approximation B which is the simplest systematic approximation to the summary theory which at least qualitatively describes the shimmy phenomena throughout the complete range of rolling velocity. The theoretical and experimental frequencies are seen to be in fairly good agreement. The theoretical and experimental divergences are in fair qualitative agreement, but quantitatively, the experimental variation is sometimes considerably below the corresponding theoretical one. This quantitative disagreement may be due to several factors. First, hysteresis effects, which may be of some importance for these data, are neglected in the theoretical calculations. A second partial explanation for the indicated disagreement arises from the fact that the theoretical calculations may be based on insufficiently accurate values of the necessary tire parameters since Kantrowitz did not provide direct measurements of the

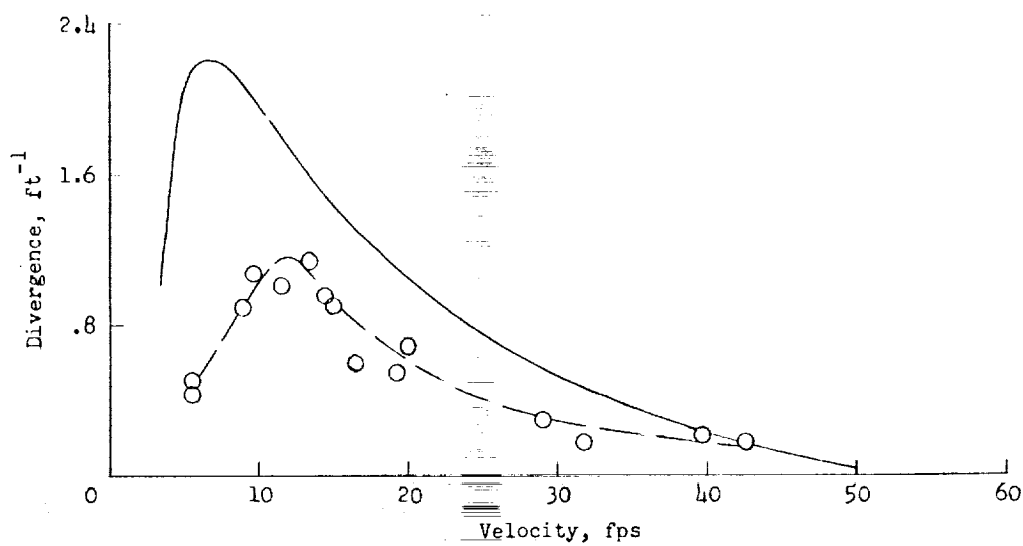
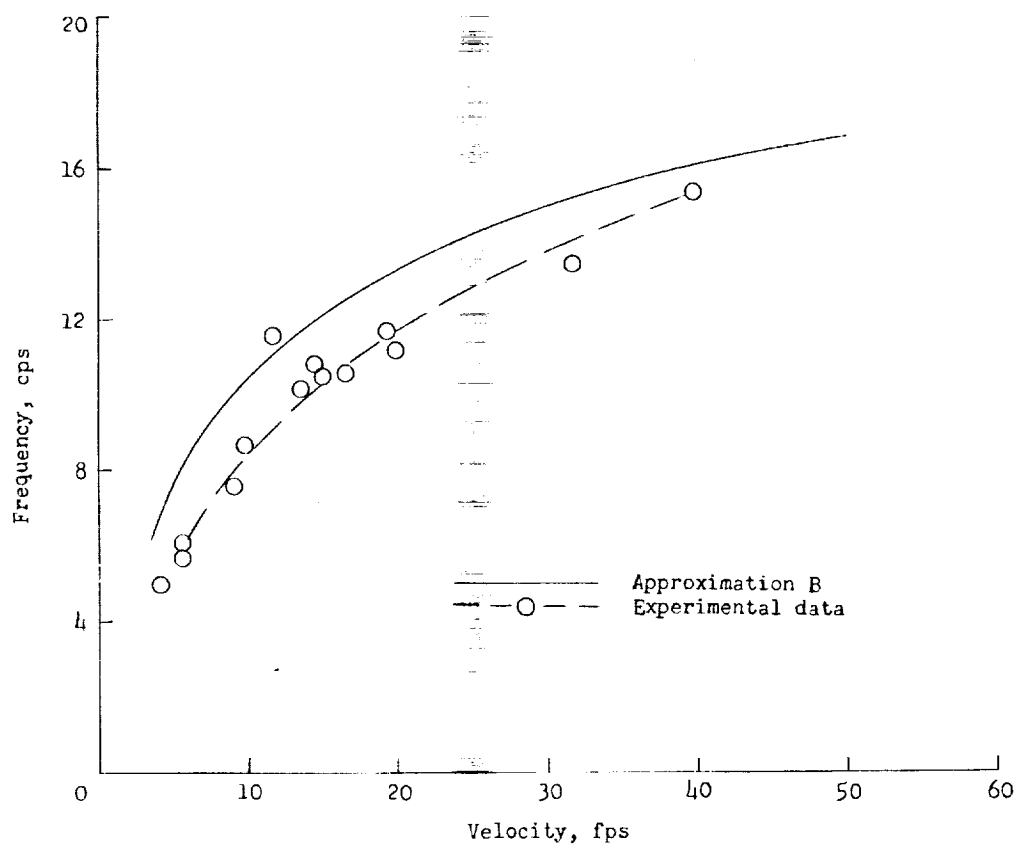


FIGURE 15

COMPARISON OF THEORETICAL AND EXPERIMENTAL SHIMMY FREQUENCY AND DIVERGENCE FOR KANTROWITZ'S EXPERIMENTAL DATA. $\kappa = 5^\circ$

most fundamental tire parameters, such as h , a , N , K_a , etc.; instead he measured only certain different secondary tire parameters. Specifically Kantrowitz measured only the quantity L , a quantity approximately equal to $aN \cos \kappa + K_a \cos^2 \kappa$ for 2 values of κ , and the path frequency v_1 and trail a for kinematic shimmy (shimmy with velocity approaching zero). The basic tire parameters used for calculating the theoretical curves in Figure 15 were approximately deduced from these quantities as follows. The quantity h was obtained from equation A-1 of the Appendix after setting $v = 0$ and substituting Kantrowitz's experimental values of L , v_1 and a for kinematic shimmy. This procedure for determining the quantity h is, however, not necessarily too accurate since equation A-1 neglects tire hysteresis effects which may be important for the condition of kinematic shimmy. The tire deflection, needed for calculating the trail, was estimated from Figure 8 of Rotta's paper.⁹⁰ The trail was computed from the tire radius, the tire deflection and the inclination. With the aid of this estimated value of trail, the tire parameters N and K_a can be obtained from Kantrowitz's approximate expressions for $aN \cos \kappa + K_a \cos^2 \kappa$. While the just discussed procedure for deducing the fundamental tire parameters for Kantrowitz's

⁹⁰ J. Rotta, op. cit.

data will probably give roughly correct values of most of the fundamental tire constants, it is believed that the limitations of this procedure and the neglect of the hysteresis effects in the theoretical calculations are sufficient reasons to prohibit the making of any strong point out of the discrepancies between theory and experiment in Figure 15. Thus, to summarize, it appears that Kantrowitz's data furnish only a rough overall confirmation of the summary theory and while quantitative agreement is poorer than for most of the previously discussed experimental data, this poorer agreement is not necessarily significant.

This completes the discussion of Case I with respect to the summary theory and its systematic approximations. Next, attention will be directed to a discussion of Case I with respect to the predictions of some of the previously published theories.

Discussion of predictions of some of the previously published theories.- Some interesting features of the previously published theories in relation to Case I are as follows.

The theory of Schlippe and Dietrich⁹¹ gives predictions which are substantially the same as the predictions of

⁹¹ B. von Schlippe and R. Dietrich, Zur Mechanik des Luftreifens, op. cit.

the summary theory as can be seen by a comparison of the predictions of these two theories in Figures 9 and 10 for Schlippe and Dietrich's model test conditions. In comparing these two theories, it should be noted that the only difference in these two sets of theoretical curves rises from a slight difference in the expressions used for the tire elastic forces and moments (see Chapter III). While the Schlippe-Dietrich theory also provides for some tire width effects, these effects for the present test conditions are believed to be relatively small and were not taken into account in computing the theoretical curves in Figures 9 and 10. From these Figures, it is seen that the differences between the stability boundaries and frequency curves for the Schlippe-Dietrich theory and the summary theory are usually small beside the differences between the theoretical curves and the experimental data. Thus, it seems reasonable to conclude that there is no great significant difference between the main features of the summary theory and the Schlippe-Dietrich theory.

Bourcier de Carbon's advanced theory⁹² provides essentially the same predictions as approximation B and will thus probably give a reasonable prediction of shimmy behavior for the complete velocity range. Similarly, Bourcier de Carbon's

⁹² Christian Bourcier de Carbon, op. cit.

elementary theory,⁹³ corresponding to approximation C2, will probably give reasonable predictions for the high velocity range.

Melzer's theory⁹⁴ correctly predicts the existence of the large trail stability boundary given by the equation $a_c = l_1$ but it also predicts the existence of stable motion in the small negative trail region between zero trail and a trail equal to $-e = -K_a/N$. This latter prediction is in disagreement with the experimental data of Schlippe and Dietrich⁹⁵ who conducted some tests in this trail range and found the motion there to be unstable.

The stability boundary according to Moreland's advanced theory⁹⁶ for the case of no damping or spring restoring forces is given by the equation

$$v_c \sqrt{\frac{I}{N l_1^3}} = \frac{1}{\sqrt{r_2}} \left(\frac{1}{1 - r_2 a_c / l_1} - \frac{a_c}{l_1} \right) \quad (7.19)$$

⁹³ Ibid.

⁹⁴ M. Melzer, op. cit.

⁹⁵ B. von Schlippe and R. Dietrich, "Das Flattern eines mit Luftreifen versehenen Rades," op. cit.

⁹⁶ William J. Moreland, "The Story of Shimmy," op. cit.

where

$$\tau_2 = N l_1 C_1^2 / I_v$$

This equation is plotted in Figure 16 for zero time constant (for which case Moreland's theory reduces to the subcase of approximation C2 where $\epsilon = K_a = 0$) and for several finite values of the time constant parameter τ_2 . It is seen that if the time constant parameter τ_2 is large there no longer exists a large trail stability boundary at the trail $a_c = l_1$. Since the actual existence of this large trail stability boundary has already been demonstrated in previous parts of this paper, it appears likely that τ_2 cannot be very large. On the other hand, if τ_2 is small, the introduction of the time constant term is seen to produce an almost linear decrease of critical trail with increasing velocity until a certain limiting velocity (equal to l_1/C_1) is reached; above this velocity, all-motion is stable. Thus, the influence of the time lag constant term is somewhat like that of the previously discussed gyroscopic moment due to tire distortion which may also produce stability at high velocities. However, in regard to the general shape of the critical trail-velocity curve, the variation predicted by consideration of the gyroscopic effect (see solid line in Figure 14) appears more like that of the experimental data

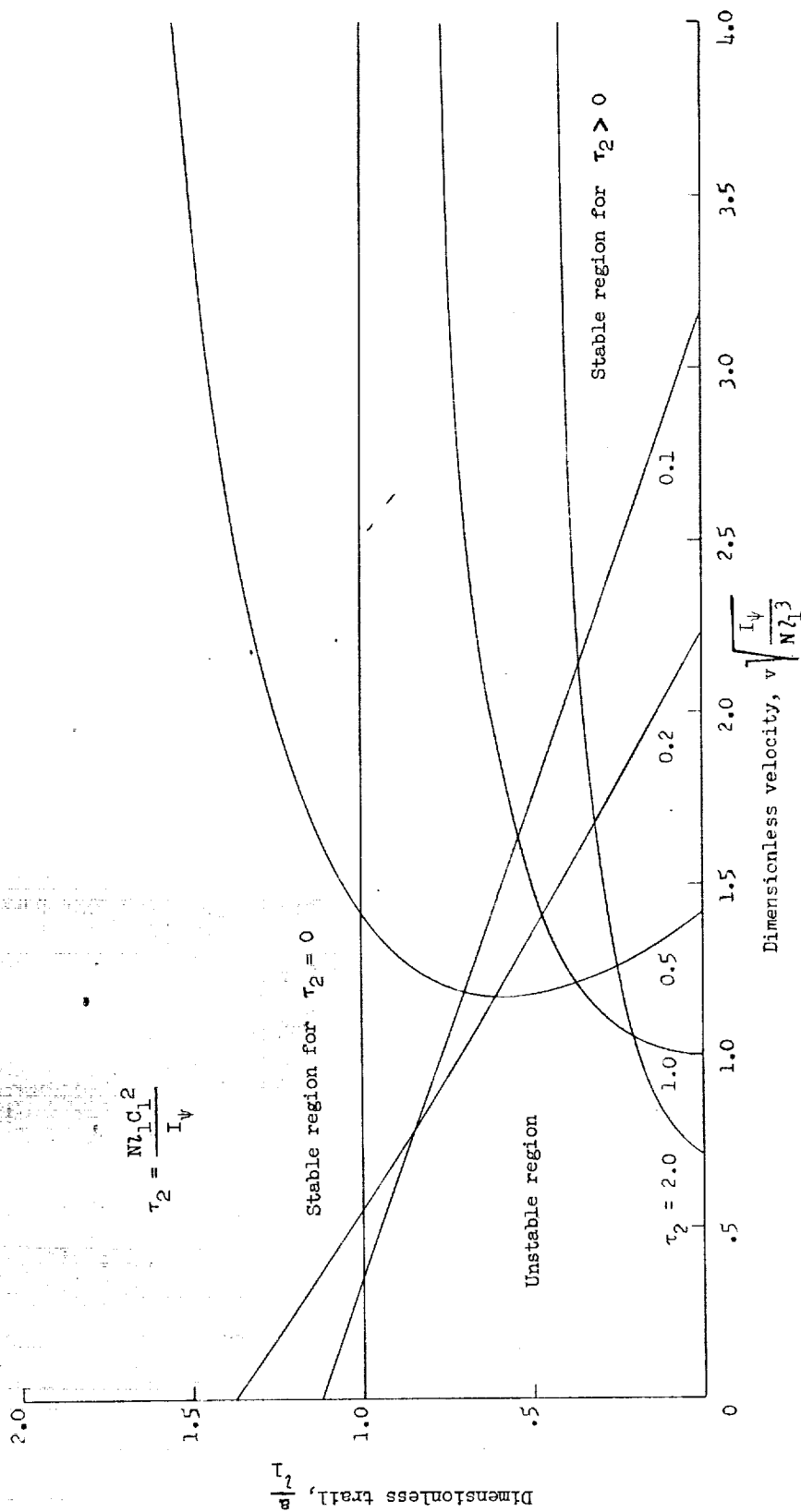


FIGURE 16
 VARIATION OF CRITICAL TRAIL WITH ROLLING VELOCITY
 ACCORDING TO MORELAND'S ADVANCED THEORY

(Figures 7 or 14) than does the nearly linear variation predicted from Moreland's time lag term for small τ_2 .

(It should be noted, however, that this criticism of Moreland's theory is based on the assumption that the time lag constant C_1 is a pure tire constant, independent of the landing gear geometrical and inertia properties. If on the other hand, Moreland considers the time lag constant to be an overall landing gear parameter, than C_1 may be a function of trail and the preceding discussion based on the assumption that C_1 is constant may be invalid.)

Moreland's elementary theory,⁹⁷ Temple's elementary theory⁹⁸ and Maier's⁹⁹ and Taylor's¹⁰⁰ theories are too crude to give any details for Case I.

Kantrowitz's theory¹⁰¹ incorrectly predicts instability for all positive trails in the absence of damping or gyroscopic moments.

⁹⁷ William J. Moreland, "Landing-Gear Vibration," op. cit.

⁹⁸ G.. Temple, "Preliminary Report on the Theory of Shimmy in Aeroplane Nose Wheels and Tail Wheels," op. cit.

⁹⁹ E. Maier, op. cit.

¹⁰⁰ J. Lockwood Taylor, op. cit.

¹⁰¹ Arthur Kantrowitz, op. cit.

Wylie's theory¹⁰² correctly predicts the existence of stability at large trails; however the particular value of critical trail predicted is given by the equation

$$a_c(a_c + \epsilon)Nl_2 = Iv^2L \quad (7.20)$$

for $\kappa = 0$. This relation implies that the critical trail is a continuously increasing function of velocity while the previously discussed experimental data clearly indicate that the critical trail rapidly reaches the maximum value l_1 .

Case II

The present section of this paper is concerned with the discussion of an idealized landing gear whose configuration is shown in Figure 6. This landing gear consists of a wheel free to swivel about an uninclined always vertical swivel axis, this swivel axis being attached by a horizontal linear spring, of spring constant k , to the supporting structure. This Case II configuration is discussed here for two reasons; first, because it gives an illustration of the effect of structural elasticity on wheel shimmy and, second, because it provides an opportunity better suited than Case I for evaluating approximations D1, D2 and D3 with respect to

¹⁰²

Jean Wylie, op. cit.

the application of these theories to landing gear problems involving structural elasticity. (It may be recalled that these three approximations were of little value in dealing with the case of a rigid landing gear strut (Case I); however, for the present case of a flexible strut, these approximations may sometimes be of value.) In discussing Case II, no further mention will be made regarding the previously published theories or of the question of agreement between theory and experiment; all discussion will be restricted to the summary theory and its systematic approximations.

The discussion of Case II proceeds as follows. First, the equations of motion for this case are derived according to the summary theory. As for the previous Case I, it is more convenient to rederive these equations of motion in a slightly different manner rather than to apply the earlier derived equations for the completely general case. After making these derivations the equations for the stability boundaries are established. Finally some curves of the damping required to prevent shimmy, as functions of strut stiffness and rolling velocity, are presented for a specific sample landing gear configuration according to the predictions of approximations C, D1, D2 and D3. (For the present case, approximations C1 and C2 are identical and are, for convenience, henceforth referred to collectively as approximation C.)

These curves are utilized to obtain some insight into the relative accuracies of the predictions of approximations D1, D2, and D3 with respect to the more advanced approximation C.

General derivation. - The derivation of the equation of motion for the summary theory proceeds as follows. The details of the landing gear considered are illustrated in Figure 6. This gear has a rigid symmetrical swiveling part having a mass m and a moment of inertia about its center of gravity I_0 . The nonswiveling part of the landing gear consists of a spring of stiffness k with an attached mass m_1 . The lateral displacement of the swivel axis is designated as η_a .

Setting the sum of the lateral spring and inertia forces acting on the swiveling part equal to the inertia reaction of its center of gravity $mD_t^2(\eta_a - c_2\theta)$ yields the relation

$$K_\lambda \lambda_0 - k\eta_a - m_1 D_t^2 \eta_a = m D_t^2 \eta_a - m c_2 D_t^2 \theta \quad (7.21)$$

and substitution for λ_0 from the relation

$$\lambda_0 = y_0 - \eta_0 = y_0 - \eta_a + a\theta \quad (7.22)$$

(see Figure 6) yields after rearrangement

$$K_{\lambda} y_0 - (m_1 D_t^2 + m D_t^2 + K_{\lambda} + k) \eta_a + (m c_2 D_t^2 + a K_{\lambda}) \theta = 0 \quad (7.23)$$

Setting the sum of the moments about the center of gravity of the swiveling part equal to the inertia reaction yields the result

$$K_a a - K_{\lambda} \lambda_0 c_1 - k \eta_a c_2 - m_1 D_t^2 \eta_a c_2 - g D_t \theta - \rho \theta - \tau v D_t \lambda_0 = I_0 D_t^2 \theta \quad (7.24)$$

(see Figure 6) where I_0 represents the moment of inertia of the swiveling structure at its center of gravity ($I_0 = I_v - m c_2^2$). Substitution for a and λ_0 according to equations (3.4) and (7.22) then, after rearrangement, yields the result

$$(\tau v D_t - K_a v^{-1} D_t + K_{\lambda} c_1) y_0 + (m_1 c_2 D_t^2 - \tau v D_t + k c_2 - K_{\lambda} c_1) \eta_a + (I_0 D_t^2 + g D_t + \tau a v D_t + \rho + K_a + a c_1 K_{\lambda}) \theta = 0 \quad (7.25)$$

The third equation for this system for the general case is given by the kinematic relation of equation (2.20). This relation, after omitting γ , replacing space derivatives by time derivatives and setting $\eta_0 = \eta_a - a\theta$, reads

$$-(1 + l_1 v^{-1} D_t + l_2 v^{-2} D_t^2 + \dots) y_0 + (l_1 - a) \theta + \eta_a = 0 \quad (7.26a)$$

or

$$-(1 + L v^{-1} D_t) e^{h v^{-1} D_t} y_0 + (l_1 - a) \theta + \eta_a = 0 \quad (7.26b)$$

The three equations (7.23), (7.25) and (7.26) completely describe the motion of the landing gear according to the general theory in terms of the three variables y_0 , η_a and θ . The corresponding equations for the systematic approximations can be easily obtained in a similar manner.

Stability boundaries.— The stability boundaries for Case II are obtained in the same manner as was indicated in the discussion of Case I. For the summary theory, they are obtained as follows.

Purely oscillatory boundaries: The equations for the purely oscillatory motion boundaries are obtained by substituting into the differential equations the expressions

$$\left. \begin{aligned} \theta &= \theta_m e^{i \nu t} \\ \eta_a &= \eta_{am} e^{i(\nu t + \sigma_1)} = \eta_{am} e^{i \nu t} (\cos \sigma_1 + i \sin \sigma_1) \\ y_0 &= y_{0m} e^{i(\nu t + \sigma_2)} = y_{0m} e^{i \nu t} (\cos \sigma_2 + i \sin \sigma_2) \end{aligned} \right\} (7.27)$$

Substitution of these relations into equations (7.23), (7.25) and (7.26b), differentiation and cancellation of $e^{i\omega t}$ and separation of real and imaginary parts into separate equations yields the expressions

$$K_{\lambda}(y_{0m} \cos \sigma_2) + (m_1 v^2 + m v^2 - K_{\lambda} - k)(\eta_{am} \cos \sigma_1) + (aK_{\lambda} - mc_2 v^2)\theta_m = 0 \quad (7.28)$$

$$K_{\lambda}(y_{0m} \sin \sigma_2) + (m_1 v^2 + m v^2 - K_{\lambda} - k)(\eta_{am} \sin \sigma_1) = 0 \quad (7.29)$$

from equation (7.23),

$$c_1 K_{\lambda}(y_{0m} \cos \sigma_2) - (\tau v v - K_{\alpha} v^{-1} v)(y_{0m} \sin \sigma_2) + (-m_1 c_2 v^2 + c_2 k - c_1 K_{\lambda})(\eta_{am} \cos \sigma_1) + \tau v v(\eta_{am} \sin \sigma_1) + (-I_0 v^2 + \rho + K_{\alpha} + a c_1 K_{\lambda})\theta_m = 0 \quad (7.30)$$

$$c_1 K_{\lambda}(y_{0m} \sin \sigma_2) + (\tau v v - K_{\alpha} v^{-1} v)(y_{0m} \cos \sigma_2) + (-mc_2 v^2 + kc_2 - K_{\lambda} c_1)(\eta_{am} \sin \sigma_1) - \tau v v(\eta_{am} \cos \sigma_1) + (g v + \tau a v v)\theta_m = 0 \quad (7.31)$$

from equation (7.25), and

$$\begin{aligned}
 & -p_{1\infty}(y_{0m} \cos \sigma_2) + p_{2\infty}(y_{0m} \sin \sigma_2) + (l_1 - a)\theta_m + \\
 & (\eta_{am} \cos \sigma_1) = 0
 \end{aligned} \tag{7.32}$$

$$\begin{aligned}
 & -p_{2\infty}(y_{0m} \cos \sigma_2) - p_{1\infty}(y_{0m} \sin \sigma_2) + \eta_{am} \sin \sigma_1 = 0
 \end{aligned} \tag{7.33}$$

from equation (7.26b). Equations (7.28) to (7.33) can be considered as six linear simultaneous algebraic equations with no constant terms in the five variables $y_{0m} \cos \sigma_2$, $y_{0m} \sin \sigma_2$, $\eta_{am} \cos \sigma_1$, $\eta_{am} \sin \sigma_1$ and θ_m . Then for this system of equations to have solutions other than zero, it is necessary that the determinant of the coefficients of any group of five of these six equations should equal zero. The determinant for equations (7.28), (7.29), (7.31), (7.32) and (7.33) reads

$c_1 K_\lambda$	$- \tau_{11} + K_\alpha v^{-1} v$	$- m_1 c_2 v^2 + c_2 k - c_1 K_\lambda$	τ_{11}	$- I_0 v^2 + \rho + K_\alpha + a c_1 K_\lambda$	
K_λ	0	$(m_1 + m) v^2 - K_\lambda - k$	0	$a K_\lambda - m c_2 v^2$	
0	K_λ	0	$(m_1 + m) v^2 - K_\lambda - k$	0	= 0 (7.34)
$-p_{1\infty}$	$p_{2\infty}$	1	0	$z_1 - a$	
$-p_{2\infty}$	$-p_{1\infty}$	0	1	0	

and for equations (7.28), (7.29), (7.30), (7.32), and (7.33) reads

$\tau_{11} - K_\alpha v^{-1} v$	$c_1 K_\lambda$	$- \tau_{11}$	$- m c_2 v^2 + k c_2 - K_\lambda c_1$	$g v + \tau_{11} v$	
K_λ	0	$(m_1 + m) v^2 - K_\lambda - k$	0	$a K_\lambda - m c_2 v^2$	
0	K_λ	0	$(m_1 + m) v^2 - K_\lambda - k$	0	= 0 (7.35)
$-p_{1\infty}$	$p_{2\infty}$	1	0	$z_1 - a$	
$-p_{2\infty}$	$-p_{1\infty}$	0	1	0	

Equation (7.34) gives the frequency for purely oscillatory motion as a function of the landing gear properties and equation (7.35) gives the amount of damping required for this purely oscillatory motion as a function of the frequency and the landing gear properties. The corresponding equations for the systematic approximations can be obtained either by following through a similar derivation for each approximation or, in some cases, by applying appropriate simplifications to equations (7.34) and (7.35). For example, to obtain the boundary equations for approximation B, whose basic equation is $l_n = 0$ for $n > 2$, p_1 and p_2 in equations (7.34) and (7.35) may be replaced by their respective series expansion expressions according to equation (5.2a) and then the appropriate higher order terms in the two series may be omitted.

Purely uniform motion: For purely uniform motion, all variables will have constant values which may be represented as

$$\theta = \theta_m$$

$$\eta_a = \eta_{am}$$

$$y_0 = y_{0m}$$

Substitution of these relations into equations (7.23), (7.25) and (7.26) yields the results

$$aK_{\lambda}\theta_m - (K_{\lambda} + k)\eta_{am} + K_{\lambda}y_{Om} = 0$$

$$(\rho + K_{\alpha} + ac_1K_{\lambda})\theta_m + (c_2k - c_1K_{\lambda})\eta_{am} + c_1K_{\lambda}y_{Om} = 0$$

$$(l_1 - a)\theta_m + \eta_{am} - y_{Om} = 0$$

For nonzero solutions of these three equations, the determinant of the coefficients of θ_m , η_{am} and y_{Om} must be zero. Evaluation of this determinant gives simply

$$a + \epsilon + \rho/N = 0 \quad (7.36)$$

Evaluation of approximations D1, D2 and D3.- In the earlier discussion of Case I, it was not possible to present a fair relative evaluation of the three parallel approximate theories D1, D2 and D3 since for Case I, none of these theories provides any realistic information. However, for the present case II, such a comparison can be made between the predictions of these three approximations and the more accurate approximation C, and a specific example is discussed here for a sample landing gear configuration having the

relative dimensions and properties: $L = 0.8r$, $h = a = 0.5r$, $c_1 = c_2 = 0.25r$, $\epsilon = 0.3r$, $m_1 = 0.35m$, $I_0 = mr^2$ and $\tau = \rho = 0$. The actual calculated behavior of this landing gear in terms of damping required for stability as a function of rolling velocity according to approximation C is shown in Figure 17 for four values of the ratio of strut stiffness to tire stiffness k/K_λ . It is seen from this Figure that as the stiffness of the strut is decreased from infinity, the damping requirement is increased. Also for large strut stiffness, the region of maximum damping required lies at low speeds while for small strut stiffness, it lies at higher speeds.

The theoretic predictions of the three theories D1, D2 and D3 for this sample landing gear are compared with the corresponding predictions of the more accurate approximation C (from Figure 17) in Figure 18 for three values of strut stiffness, $k = 0.2K_\lambda$, $1.0K_\lambda$, and $5.0K_\lambda$. It is seen that for each strut stiffness, approximations D2 and D3 provide a considerable overestimate of the damping required for stability. On the other hand, approximation D1 gives results in good agreement with those of approximation C for the ratios $k/K_\lambda = 0.2$ and 1.0 but this approximation greatly underestimates the damping for the large value of strut stiffness $k/K_\lambda = 5.0$.

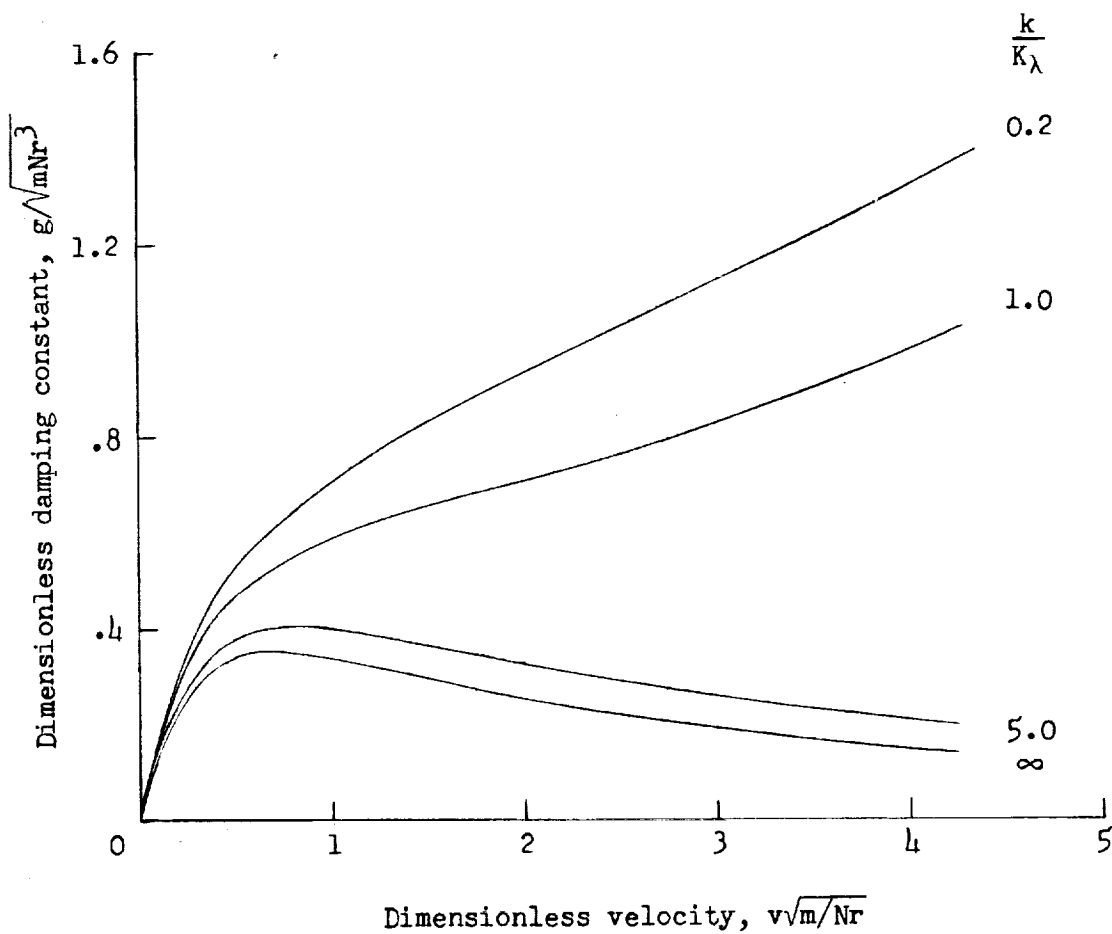


FIGURE 17

INFLUENCE OF STRUT STIFFNESS ON DAMPING REQUIRED FOR STABILITY
 ACCORDING TO APPROXIMATION C FOR A SAMPLE LANDING GEAR HAVING
 THE PROPERTIES $L = 0.8r$, $h = a = 0.5r$, $c_1 = c_2 = 0.25r$,
 $\epsilon = 0.3r$, $m_1 = 0.35m$, $I_0 = mr^2$, AND $\tau = \rho = 0$

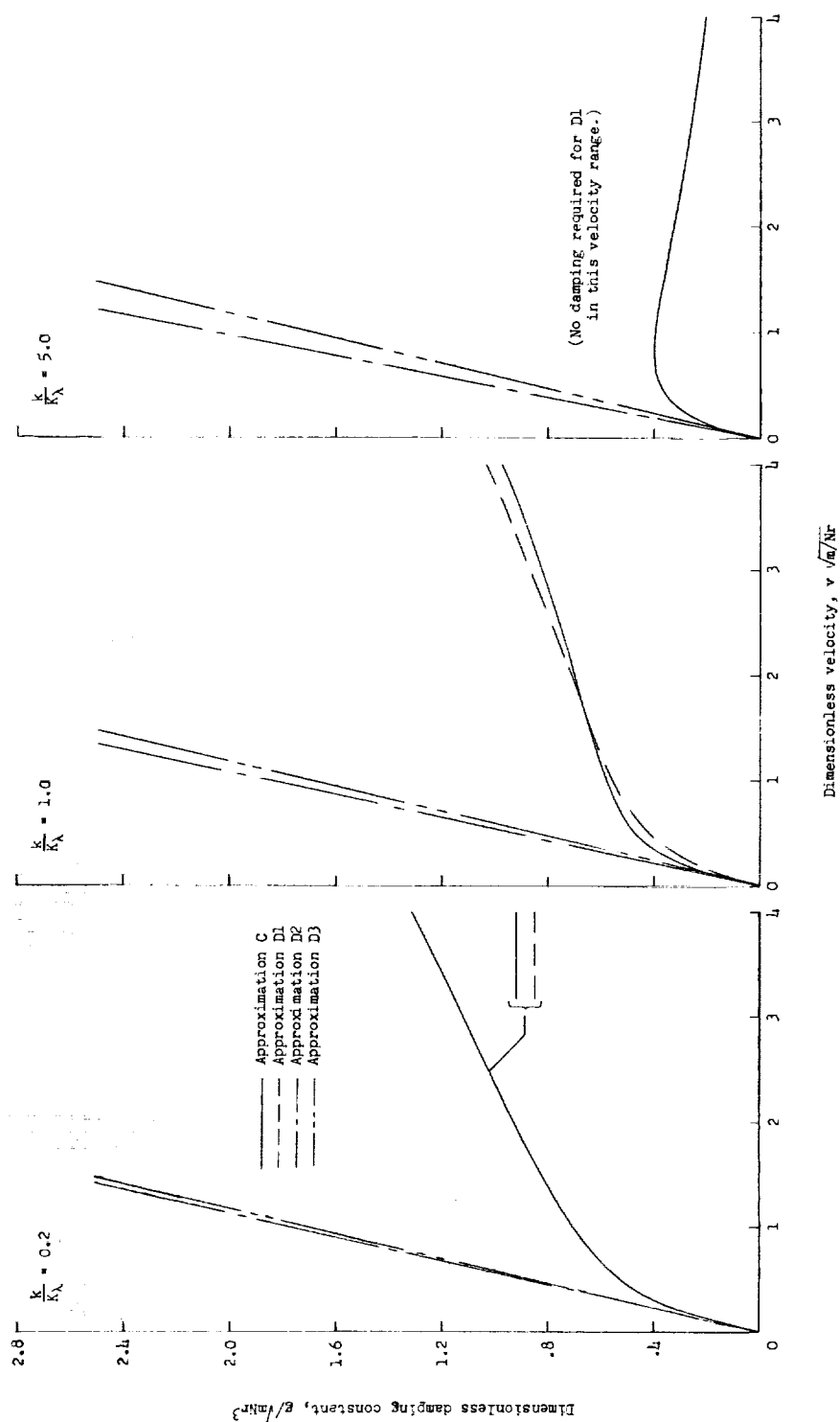


FIGURE 18

COMPARISON OF DAMPING REQUIRED FOR STABILITY ACCORDING TO APPROXIMATION C, D1, D2, AND D3 FOR A SAMPLE LANDING GEAR HAVING THE PROPERTIES

$$L = 0.8r, h = a = 0.5r, c_1 = c_2 = 0.25r, \epsilon = 0.3r,$$

$$m_1 = 0.35m, I_0 = mr^2, \text{ AND } \tau = \rho = 0$$

In view of these comparisons, it appears that approximations D2 and D3 will not, in general, give reliable quantitative estimates of the damping required for stability. For approximation D1, it appears that this theory may give reasonable results for some cases where the lateral stiffness of the strut does not greatly exceed the lateral stiffness of the tire. This latter conclusion is, of course, not necessarily a general conclusion since it is based on only one set of landing gear parameters. To determine the degree to which it is valid in general would require a more extensive investigation for a range of landing gear properties.

CHAPTER VIII

SUMMARY

Over the past 25 years, a large number of at least slightly different theories of tire motion and wheel shimmy have been developed but there has not been much effort directed to the reconciliation of these different theories. The present paper provides this needed correlation by demonstrating that most of the existing theories represent varying degrees of approximation to a general summary theory developed herein which is a minor modification of the basic theory of Schlippe and Dietrich. In most cases where strong differences exist between the existing theories and the summary theory, the existing theories are shown to be of inferior merit.

A series of systematic approximations to the summary theory is developed for the treatment of problems too simple to require the complexity of the complete summary theory.

Comparisons of the existing experimental data with the predictions of the summary and systematic approximation theories provide a fair substantiation of the higher approximate theories. However, some discrepancies do exist which may be due to tire hysteresis effects or other unknown influences. Further work may be required to resolve these discrepancies.

Langley

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APPENDIX

APPENDIX

STABILITY BOUNDARIES FOR CASE I

The following equations describe the conditions for which purely oscillatory motion is possible for Case I for the summary theory and the systematic approximations.

For the summary theory and approximations A to C2

$$v^2 = \frac{(a^2 K_\lambda + K_a \cos^2 \kappa + \rho + \rho_\kappa)(p_1^2 + p_2^2)}{I_\downarrow v_1^2(p_1^2 + p_2^2) - \tau v_1 p_2 (\sigma l_1 \cos \kappa - a) \cos \kappa} +$$

$$\frac{[(a K_\lambda + c_\lambda F_z \sin \kappa) p_1 - v_1 p_2 K_a \cos \kappa](\sigma l_1 \cos \kappa - a)}{I_\downarrow v_1^2(p_1^2 + p_2^2) - \tau v_1 p_2 (\sigma l_1 \cos \kappa - a) \cos \kappa}$$

(A-1)

and

$$g = \frac{(\sigma l_1 \cos \kappa - a)}{v_1 v (p_1^2 + p_2^2)} \left[p_2 (a K_\lambda + c_\lambda F_z \sin \kappa) + \right.$$

$$\left. v_1 p_1 (K_a \cos \kappa - \tau v^2 \cos \kappa) \right] - a \tau v \cos \kappa \quad (A-2)$$

where for the summary theory

$$p_1 = p_{1\bullet} = \cos v_1 h - L v_1 \sin v_1 h$$

$$p_2 = p_{2\bullet} = \sin v_1 h + L v_1 \cos v_1 h$$

for approximation A

$$p_1 = 1 - l_2 v_1^2$$

$$p_2 = l_1 v_1 - l_3 v_1^3$$

for approximation B

$$p_1 = 1 - l_2 v_1^2$$

$$p_2 = l_1 v_1$$

and for approximations C1 and C2

$$p_1 = 1$$

$$p_2 = l_1 v_1$$

For approximations D1 and D3, purely oscillatory motion does not exist.

For approximation D2

$$v^2 = (a^2 K_\lambda + a \epsilon K_\lambda \cos \kappa + \rho + \rho_\kappa + \tau v^2 \cos^2 \kappa) / I_\downarrow$$

$$g = v \left[\frac{I_{\downarrow} (aK_{\lambda} \cos \kappa + \epsilon K_{\lambda} \cos^2 \kappa + \sigma c_{\lambda} F_z \sin \kappa \cos \kappa)}{a^2 K_{\lambda} + a \epsilon K_{\lambda} \cos \kappa + \rho + \rho_{\kappa} + \tau v^2 \cos^2 \kappa} - \right. \\ \left. a \tau \cos \kappa \right]$$

The stability boundaries for uniform motion are obtained by setting the coefficient of the y_0 terms in the various differential equations equal to zero. For example, for the summary theory and approximations A and B, the equation

$$\sigma a N \cos \kappa + K_{\lambda} \cos^2 \kappa + \rho + \rho_{\kappa} + u_{\kappa} = 0$$

describes this stability boundary.